Far Eastern Mathematical Journal. 2017. V. 17. No 1. P. 1–9

UDC 511.42
MSC2010 11K60

© N. V. Budarina[1]

Inhomogeneous Diophantine approximation on curves with non-monotonic error function

In this paper we prove a convergent part of inhomogeneous Groshev type theorem for non–degenerate curves in Euclidean space where an error function is not necessarily monotonic. Our result naturally incorporates and generalizes the homogeneous measure theorem for non-degenerate curves. In particular, the method of Inhomogeneous Transference Principle and Sprindzuk's method of essential and inessential domains are used in the proof.

Key words: *Inhomogeneous Diophantine approximation, Khintchine theorem, non-degenerate curve.*

Introduction and Statements

In 1998 Kleinbock and Margulis [1] established the Baker–Sprindzuk conjecture concerning homogeneous Diophantine approximation on manifolds. An inhomogeneous version was then proved by Beresnevich and Velani [2]. The theory of inhomogeneous Diophantine approximation on manifolds was started with the result of V. I. Bernik, D. Dickinson and M. Dodson [3]. The significantly stronger Groshev type theory for dual Diophantine approximation on manifolds is established in [4–6] for the homogeneous case and in [7] for the inhomogeneous case. In all of these results the error function Ψ was assumed to be monotonic. In 2005 Beresnevich [8] showed that the condition that Ψ is monotonic could be removed for the Veronese curve $\mathcal{V}_n = = \{(x, x^2, \ldots, x^n) : x \in \mathbb{R}\}$; he conjectured that the result should also hold for any non–degenerate curve in Euclidean space. This was proved in [9].

Our main result below is a convergent part of Groshev type theorem for inhomogeneous Diophantine approximation on non–degenerate curves in Euclidean space without monotonicity condition. First some notation is needed. Let $\mathcal{F}_n$ be the set of functions

$$a_n f_n(x) + \ldots + a_1 f_1(x) + a_0,$$

with $n \geq 2$, $\mathbf{a} = (a_0, \ldots, a_n) \in \mathbb{Z}^{n+1} \setminus \{0\}$, and $f_1, f_2, \ldots, f_n$ be $C^{(n)}$ functions from $\mathbb{R} \to \mathbb{R}$ with non–vanishing Wronskian $wr(f_1', \ldots, f_n')(x)$ almost everywhere. For $F \in \mathcal{F}_n$

[1] Institute for Applied Mathematics, Khabarovsk Division, Far-Eastern Branch of the Russian Academy of Sciences, Khabarovsk, Russia
E-mail: buda77@mail.ru (N. V. Budarina).

define the height of F as $H = H(F) = \max_{0\le j\le n} |a_j|$. The Lebesgue measure of a measurable set $A \subset \mathbb{R}$ is denoted by $\mu(A)$.

Define a real valued function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ and a function $\theta : \mathbb{R} \to \mathbb{R}$. Denote by $\mathcal{L}_{n,\theta}(\Psi)$ the set of $x \in \mathbb{R}$ such that the inequality

$$|F(x) + \theta(x)| < \Psi(H(F)) \tag{1}$$

has infinitely many solutions $F \in \mathcal{F}_n$.

The main result of this paper is the following statement.

Theorem 1. *Let $n \ge 2$ and $\theta : \mathbb{R} \to \mathbb{R}$ be a function such that $\theta \in C^{(n)}$. Let $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ be an arbitrary function (not necessarily monotonic) such that the sum $\sum\limits_{h=1}^{\infty} h^{n-1}\Psi(h)$ converges. Then $\mu(\mathcal{L}_{n,\theta}(\Psi)) = 0$.*

Throughout, the Vinogradov symbol $\ll$ is used so that if K and M are positive real numbers then $K \ll M$ means that there exists $C > 0$ such that $K \le CM$. If $K \ll M$ and $M \ll K$ we write $K \asymp M$.

1 Proof of Theorem 1

First note that since $\sum\limits_{h=1}^{\infty} h^{n-1}\Psi(h)$ converges, $h^{n-1}\Psi(h)$ tends to 0 as $h \to \infty$. Therefore,

$$\Psi(h) = o(h^{-n+1}). \tag{2}$$

The set $S = \{x \in \mathbb{R} : wr(f_1', \dots, f_n')(x) = 0\}$ is closed and of zero measure. Thus $\mathbb{R} \setminus S$ is open and therefore an F_σ set. We can write $\mathbb{R} \setminus S = \bigcup\limits_{k=1}^{\infty} [a_k, b_k]$. It is therefore sufficient to prove the theorem for a closed interval I. Also, since $|wr(f_1', \dots, f_n')(x)| \ne 0$ almost everywhere we will assume from now on, without loss of generality that

$$|wr(f_1', \dots, f_n')(x)| \ge \varepsilon = \varepsilon(I) > 0 \tag{3}$$

for all x in such an interval I. Since the functions $\mathbf{f} = (f_1, \dots, f_n)$ and θ are $C^{(n)}$ then we can assume that there exists a constant $K_0 = K_0(I, \mathbf{f}, \theta)$ such that

$$\max_{0\le i\le n} \sup_{x\in I} |\mathbf{f}^{(i)}(x)| \le K_0 \quad \text{and} \quad \max_{0\le i\le n} \sup_{x\in I} |\theta^{(i)}(x)| \le K_0. \tag{4}$$

Lemma 1 [9]. *If $|wr(f_1', \dots, f_n')(x)| \ge \varepsilon$ then $|f_i(x)f_j'(x) - f_i'(x)f_j(x)| > \frac{\varepsilon\gamma^2}{2^{n+1}n!K_0^n}$ for all i, j in $\{1, \dots, n\}$.*

From now on, it is therefore assumed without loss of generality that

$$|f_i(x)f_j'(x) - f_i'(x)f_j(x)| \ge \delta_2 = \frac{\varepsilon\gamma^2}{2^{n+1}n!K_0^n} \tag{5}$$

for all $i, j \in \{1, \dots, n\}$ with $i \ne j$.

For the proof of main result we will need some properties of the functions $F \in \mathcal{F}_n$. The following lemma is a modification and combination of Lemmas 2 and 3 of Pyartli, [11]. We are assuming that (3) holds.

Lemma 2 (Borel–Cantelli). *Let A_j be a family of Lebesgue measurable sets and let A_∞ be the set of points $x \in \mathbb{R}$ which lie in infinitely many A_j. If $\sum_{j=1}^{\infty} \mu(A_j) < \infty$ then $\mu(A_\infty) = 0$.*

1.1 The case of small derivative

Proposition 1. *Let $n \geq 2$. Then, $\mu(\mathcal{L}_1(n, \theta)) = 0$.*

Proof. First $\mathcal{L}_1(n, \theta)$ is written as a lim sup set. For $F \in \mathcal{F}_n$ define

$$B(F) = \{x \in I : |F(x) + \theta(x)| < H(F)^{-n+1}, \ |F'(x) + \theta'(x)| < H(F)^{-v}\}.$$

Then

$$\mathcal{L}_1(n, \theta) = \bigcap_{N=1}^{\infty} \bigcup_{t=N}^{\infty} \bigcup_{F \in \mathcal{F}_n^t} B(F),$$

where

$$\mathcal{F}_n^t := \{F \in \mathcal{F}_n, \ 2^t \leq H(F) < 2^{t+1}\}.$$

To prove the proposition it will be shown that a larger set (containing $\mathcal{L}_1(n, \theta)$) has measure zero and then the Inhomogeneous Transference Principle proved in [2] will be used. The Inhomogeneous Transference Principle allows the transfer of zero measure statements for homogeneous lim sup sets to inhomogeneous lim sup sets and is described below.

Inhomogeneous Transference Principle. Most of this section is adapted from [2, Case B]. For our purposes the two countable indexing sets $\mathbf{T}$ and $\mathcal{A}$ from [2] are the sets $\mathbf{T} = \mathbb{N} \cup \{0\}$ and $\mathcal{A} = \mathcal{F}_n$. Throughout, J denotes a finite open interval in $\mathbb{R}$ with closure denoted by $\bar{J}$. Let $\mathcal{H}$ and I be two maps from $(\mathbb{N} \cup \{0\}) \times \mathcal{F}_n \times \mathbb{R}^+$ into the set of open subsets of $\mathbb{R}$ such that

$$\mathcal{H}(t, F, \epsilon) = I_0^t(F, \epsilon), \quad I(t, F, \epsilon) = I_\theta^t(F, \epsilon).$$

For the specific case considered in this article the sets $I_0^t(F, \epsilon)$ and $I_\theta^t(F, \epsilon)$ are defined as follows:

$$I_\theta^t(F, \epsilon) = \begin{cases} \{x \in I : \ |F(x) + \theta(x)| < 2^{t(-n+1)}\epsilon, \ |F'(x) + \theta'(x)| < 2^{-tv}\epsilon\} & \text{if } F \in \mathcal{F}_n^t, \\ \emptyset & \text{else;} \end{cases}$$

and

$$I_0^t(F, \epsilon) = \begin{cases} \{x \in I : \ |F(x)| < 2^{t(-n+1)}\epsilon, \ |F'(x)| < 2^{-tv}\epsilon\} & \text{if } F \in \bigcup_{s=0}^{t+1} \mathcal{F}_n^s \\ \emptyset & \text{else.} \end{cases} \tag{6}$$

Let $\delta \in \mathbb{R}$ and define the function $\phi_\delta(t) = 2^{\delta t}$. Also, define $\Phi = \{\phi_\delta : 0 \leq \delta < v/2\}$. For any $\phi \in \Phi$ define

$$I_\theta^t(\phi) = \bigcup_{F \in \mathcal{F}_n} I_\theta^t(F, \phi(t)) = \bigcup_{F \in \mathcal{F}_n^t} I_\theta^t(F, \phi(t))$$

and denote by $\Lambda_I(\phi)$ the limsup set

$$\Lambda_I(\phi) = \bigcap_{N=1}^{\infty} \bigcup_{t=N}^{\infty} I_\theta^t(\phi).$$

Intersection Property: Let Φ denote a set of functions $\phi : \mathbb{N} \cup \{0\} \to \mathbb{R}^+$. The triple $(\mathcal{H}, I, \Phi)$ is said to satisfy the intersection property if for any $\phi \in \Phi$ there exists $\phi^* \in \Phi$ such that for all but finitely many $t \in \mathbb{N} \cup \{0\}$ and all distinct $F, \tilde{F} \in \mathcal{F}_n$

$$I_\theta^t(F, \phi(t)) \cap I_\theta^t(\tilde{F}, \phi(t)) \subset I_0^t(\phi^*). \tag{7}$$

Contracting Property: Let $\{k_t\}_{t\in\mathbb{N}}$ be a sequence of positive numbers such that

$$\sum_{t\in\mathbb{N}\cup\{0\}} k_t < \infty. \tag{8}$$

The measure μ is said to be *contracting with respect to* (I, Φ) if for any $\phi \in \Phi$ there exists $\phi^+ \in \Phi$ such that for all but finitely many t and all $F \in \mathcal{F}_n$ there exists a collection $C_{t,F}$ of balls B centred in $\bar{J}$ satisfying the following three conditions:

$$\bar{J} \cap I_\theta^t(F, \phi(t)) \subset \bigcup_{B\in C_{t,F}} B, \tag{9}$$

$$\bar{J} \bigcap \bigcup_{B\in C_{t,F}} B \subset I_\theta^t(F, \phi^+(t)), \tag{10}$$

$$\mu(5B \cap I_\theta^t(F, \phi(t))) \le k_t \mu(5B). \tag{11}$$

We now state the theorem from [2].

Theorem 2 (Inhomogeneous Transference Principle). *Suppose that $(\mathcal{H}, I, \Phi)$ satisfies the intersection property and that μ is contracting with respect to (I, Φ). If, for all $\phi \in \Phi$, $\mu(\Lambda_{\mathcal{H}}(\phi)) = 0$ then for all $\phi \in \Phi$, $\mu(\Lambda_I(\phi)) = 0$.*

First the contracting and intersection properties are verified and then it will be shown that $\mu(\Lambda_{\mathcal{H}}(\phi_\delta)) = 0$. This will imply using the transference principle that $\Lambda_I(\phi_\delta)$ has measure zero and further that $\mu(\mathcal{L}_1(n, d)) = 0$ as required.

1.1.1 Verifying the intersection property

Let $t \in \mathbb{N} \cup \{0\}$ and $F, \tilde{F} \in \mathcal{F}_n$ with $F \ne \tilde{F}$. Suppose that

$$x \in I_\theta^t(F, \phi_\delta(t)) \cap I_\theta^t(\tilde{F}, \phi_\delta(t)).$$

Then, the inequalities

$$|F(x) + \theta(x)| < \phi_\delta(t)2^{t(-n+1)} \quad \text{and} \quad |\tilde{F}(x) + \theta(x)| < \phi_\delta(t)2^{t(-n+1)},$$
$$|F'(x) + \theta'(x)| < \phi_\delta(t)2^{-vt} \quad \text{and} \quad |\tilde{F}'(x) + \theta'(x)| < \phi_\delta(t)2^{-vt}$$

holds.

Let $R(x) = (F(x) + \theta(x)) - (\tilde{F}(x) + \theta(x))$. Then,

$$|R(x)| < 2\phi_\delta(t)2^{t(-n+1)} < \phi_{\delta'}(t)2^{t(-n+1)},$$
$$|R'(x)| < 2^{1-vt}\phi_\delta(t) < 2^{-vt}\phi_{\delta'}(t),$$

for all $t > \frac{1}{v/2-\delta}$ and where $\phi_{\delta'} \in \Phi$. Clearly R cannot be constant for $n \geq 2$ and $t \geq 2$, so $R \in \bigcup_{s=0}^{t+1} \mathcal{F}_n^s$. Thus, $x \in I_0^t(R, \phi_{\delta'}(t))$ and (7) is satisfied with $\phi^* = \phi_{\delta'}$.

1.1.2 Verifying the contracting property

The following definition from [1] will be used.

Definition 1. Let C and α be positive numbers and $f : I \to \mathbb{R}$ be a function defined on the open interval $I \subset \mathbb{R}$. Then f is called (C, α)–good on I if, for any open interval $B \subset I$ and any $\epsilon > 0$,

$$\mu(\{x \in B : |f(x)| < \epsilon \sup_{x\in B} |f(x)|\}) \leq C\epsilon^\alpha \mu(B).$$

Several useful facts about (C, α)-*good* functions are listed below.

Lemma 3. *([6, Lemma 3.1]) Let $I \subset \mathbb{R}$ and $C, \alpha > 0$ be given.*

(i) If f is (C, α)-good on I then so is λf for any $\lambda \in \mathbb{R}$.

(ii) If f_i, $i \in I_0$, are (C, α)-good on I then so is $\sup_{i\in I_0} |f_i|$.

(iii) If f is (C, α)-good on I and $c_1 \leq \frac{|f(x)|}{|g(x)|} \leq c_2$ for all $x \in I$, then g is $(C(c_2/c_1)^\alpha, \alpha)$-good on I.

(iv) If f is (C, α)-good on I then f is (C', α')-good on I' for every $C' \geq C$, $\alpha' \leq \alpha$ and $I' \subset I$.

Lemma 4. *[7, Corollary 3] Let U be an open subset of $\mathbb{R}^m$, $\mathbf{x}_0 \in U$ and let $\mathbf{f} = (f_1, \dots, f_n) : U \to \mathbb{R}^n$ be n-nondegenerate at $\mathbf{x}_0$ for some $n \geq 2$. Let $\theta \in C^{(n)}(U)$. Then there exists a neighborhood $V \subset U$ of $\mathbf{x}_0$ and a positive constants C and H_0 such that for any $\mathbf{a} \in \mathbb{R}^n$ satisfying $|\mathbf{a}| \geq H_0$*

(a) $a_0 + \mathbf{a} \cdot \mathbf{f} + \theta$ is $(C, \frac{1}{nm})$-good on V for every $a_0 \in \mathbb{R}$, and

(b) $|\nabla(\mathbf{a} \cdot \mathbf{f} + \theta)|$ is $(C, \frac{1}{m(n-1)})$-good on V.

Here ∇ denotes the gradient operator. Note that in the case $m = 1$ the map $\mathbf{f}$ is nondegenerate iff $wr(f_1', \dots, f_n')(x) \neq 0$ almost everywhere.

Lemma 5. *[7, Corollary 4] Let $U, \mathbf{x}_0, \mathbf{f}$ and θ be as in Lemma 4. Then for every sufficiently small neighborhood $V \subset U$ of $\mathbf{x}_0$, there exists $H_0 > 1$ such that*

$$\inf_{\substack{(\mathbf{a},a_0)\in\mathbb{R}^{n+1} \\ |\mathbf{a}|\geq H_0}} \sup_{\mathbf{x}\in V} |a_0 + \mathbf{a} \cdot \mathbf{f}(\mathbf{x}) + \theta(\mathbf{x})| > 0.$$

Since $\mathbf{F}_{t,F}$ is a $(C, \frac{1}{n})$-good on $5J$ for sufficiently large $\mathbf{t}$ it follows from (9)–(11), that

$$\begin{aligned} \mu(I_\theta^t(F, \phi_\delta(t)) \cap 5B) &\leq \mu(\{x \in 5B : \mathbf{F}_{t,F}(x) \leq 2^{-\delta^* t} \sup_{x\in 5B} \mathbf{F}_{t,F}(x)\}) \\ &\leq 2^{-\frac{\delta^* t}{n}} C\mu(5B) \end{aligned} \tag{12}$$

for sufficiently large t. This verifies (11) with $k_t := 2^{-\frac{\delta^* t}{n}} C$ and it is easily seen that the convergence condition (8) is fulfilled.

$\square$

1.2 The case of big derivative

Proposition 2. *Let $n \geq 2$. Then, $\mu(\mathcal{L}_2(n, \theta, \Psi)) = 0$.*

P r o o f. Let $\mathcal{F}_n(H) = \{F \in \mathcal{F}_n : H(F) = H\}$, then $\mathcal{F}_n = \cup_{H=1}^{\infty} \mathcal{F}_n(H)$. Now consider $F \in \mathcal{F}_n(H)$ satisfying $H^{-v} \leq |F'(x) + \theta'(x)|$. For the remaining case we need the following. The set of solutions of (1) in I consists of at most n intervals. Each of these intervals can be further divided into subintervals on which $F' + \theta'$ is also monotonic (at most $n - 1$ of them). Each of these new intervals is finally further subdivided into intervals with respect to the value of $F'(x) + \theta'(x)$. Any interval on which $|F'(x) + \theta'(x)| < H^{-v}$ has already been considered. For $F \in \mathcal{F}_n(H)$, let $I_j(F, \theta)$ be one of the remaining intervals; thus, on $I_j(F, \theta)$, $F + \theta$ and $F' + \theta'$ are monotonic and $|F(x) + \theta(x)| < \Psi(H(F))$, $H^{-v} \leq |F'(x) + \theta'(x)|$ for all $x \in I_j(F, \theta)$. The number of $I_j(F, \theta)$ is clearly finite. Let $\bar{I}_j(F, \theta)$ denote the closure of $I_j(F, \theta)$ and $\alpha_{j,F}$ denote a point in $\bar{I}_j(F, \theta)$ such that

$$|F'(\alpha_{j,F}) + \theta'(\alpha_{j,F})| = \min_{x \in \bar{I}_j(F)} |F'(x) + \theta'(x)|.$$

For convenience we will use F_θ to denote the function $F(x) + \theta(x)$.

Lemma 6. *[11] Let $a_1, a_2 > 0$. Let ψ be an n-times continuously differentiable function on (b_1, b_2) satisfying $|\psi^{(n)}(x)| \geq a_1$ for all $x \in (b_1, b_2)$. Then*

$$\mu(\{x \in (b_1, b_2) : \psi(x) < a_2\}) \leq c(n)(a_2/a_1)^{1/n}.$$

From Lemma 6 we have

$$\mu(I_j(F, \theta)) \leq c(n)\Psi(H)|F'_\theta(\alpha_{j,F})|^{-1}. \tag{13}$$

It follows from the choice of $\alpha_{j,F}$ that $H^{-v} \leq |F'_\theta(\alpha_{j,F})|$.

Now we are ready to complete the proof of Theorem 1. The three remaining cases in the proof concern different ranges for the size of $F'_\theta(\alpha_{j,F})$.

Case I. For $F \in \mathcal{F}_n(H)$, let $\sigma(F_\theta)$ be the union of intervals $I_j(F, \theta)$ for which $|F'_\theta(\alpha_j)| \geq c_1 H^{1/2}$. Hence, $\sigma(F_\theta)$ is the set of $x \in I$ which satisfy $|F_\theta(x)| < \Psi(H)$ and x lies in some interval $I_j(F, \theta)$ for which

$$|F'_\theta(\alpha_{j,F})| \geq c_1 H^{1/2}. \tag{14}$$

For every $F \in \mathcal{F}_n(H)$ and every j, where $\alpha_{j,F} \in \sigma(F_\theta)$, and some constant $c_2 = c_2(n)$ define the set $\sigma_{1,j}(F_\theta)$ of points $x \in I$ which satisfy

$$|x - \alpha_{j,F}| < c_2 |F'_\theta(\alpha_{j,F})|^{-1}$$

] for $\alpha_{j,F} \in \sigma(F_\theta)$. Let $\sigma_1(F_\theta) = \cup_j \sigma_{1,j}(F_\theta)$. From (13), for $H > H_0(c_2)$, the inequality $\sigma(F_\theta) \subset \sigma_1(F_\theta)$ holds and

$$\mu(\sigma(F_\theta)) \leq c(n) c_2^{-1} \Psi(H) \mu(\sigma_1(F_\theta)). \tag{15}$$

For each j with $\alpha_{j,F} \in \sigma(F_\theta)$ develop F as a Taylor series on $\sigma_{1,j}(F_\theta)$ so that

$$F_\theta(x) = F_\theta(\alpha_{j,F}) + F'_\theta(\alpha_{j,F})(x - \alpha_{j,F}) + F''_\theta(\xi_1)(x - \alpha_{j,F})^2/2,$$

where ξ_1 is between x and $\alpha_{j,F}$. Estimate each term in the above equation to obtain

$$\begin{gathered}|F_\theta(\alpha_{j,F})| < \Psi(H) < c_2,\\ |F'_\theta(\alpha_{j,F})(x - \alpha_{j,F})| < c_2,\\ |F''_\theta(\alpha_{j,F})(x - \alpha_{j,F})^2| < 2nK_0H(c_2|F'_\theta(\alpha_{j,F})|^{-1})^2 = 2nK_0c_2^2c_1^{-2}.\end{gathered}$$

Case II. This time, for $F \in \mathcal{F}_n(H)$ use $\sigma(F_\theta)$ to denote the union of intervals $I_j(F,\theta)$ for which $1 \leq |F'(\alpha_{j,F})| < c_1H^{1/2}$. Hence $\sigma(F_\theta)$ is the set of $x \in I$ which satisfy

$$|F_\theta(x)| < \Psi(H),$$

and x lies in some $I_j(F,\theta)$ for which

$$1 \leq |F'_\theta(\alpha_{j,F})| < c_1H^{1/2}. \tag{16}$$

Now define expansion of $I_j(F,\theta)$ as follows:

$$\sigma_{2,j}(F_\theta) := \{x \in I : dist(x, I_j(F,\theta)) < c_3H^{-1}|F'(\alpha_{j,F})|^{-1}\}, \ \ c_3 > c(n).$$

Let $\sigma_2(F_\theta) = \bigcup\limits_j \sigma_{2,j}(F_\theta)$. It is readily verified that

$$\mu(\sigma(F_\theta)) \leq c_3^{-1}c(n)H\Psi(H)\mu(\sigma_2(F_\theta)). \tag{17}$$

First, the essential intervals are investigated. Summing the measure of essential intervals gives

$$\sum_{F\in\mathcal{F}_{n,\mathbf{b}_1}(H)} \sum_{\substack{j \\ \sigma_{2,j}(F_\theta)\ \text{essential}}} \mu(\sigma_{2,j}(F_\theta)) \ll |I|.$$

From this, (17) and the fact that the number of vectors $\mathbf{b}_1$ is $\ll H^{n-2}$, we have

$$\sum_{\mathbf{b}_1} \sum_{F\in\mathcal{F}_{n,\mathbf{b}_1}(H)} \mu(\sigma(F_\theta)) \ll H^{n-1}\Psi(H)|I|.$$

Finally, we obtain

$$\sum_{H=1}^{\infty} \sum_{\mathbf{b}_1} \sum_{F\in\mathcal{F}_{n,\mathbf{b}_1}(H)} \mu(\sigma(F_\theta)) < \infty.$$

Thus, by the Borel–Cantelli Lemma, the set of points x which belong to infinitely many essential domains is of measure zero.

Case III. This is very similar to the previous case. For $F \in \mathcal{F}_n(H)$ use $\sigma(F_\theta)$ to denote the union of intervals $I_j(F,\theta)$ for which $H^{-v} \leq |F'_\theta(\alpha_{j,F})| < 1$ with $0 < v < 1/4$. Hence $\sigma(F_\theta)$ is the set of $x \in I$ which satisfy

$$|F_\theta(x)| < \Psi(H),$$

and x lies in some $I_j(F,\theta)$ for which

$$H^{-v} \le |F'_\theta(\alpha_{j,F})| < 1.$$

Fix the vector $\mathbf{b}_1$ as above and define the following expansions of $I_j(F,\theta)$:

$$\sigma_{3,j}(F_\theta) := \{x \in I : \, dist(x, I_j(F,\theta)) < c_4 H^{-1}|F'(\alpha_{j,F})|^{-1}\}, \;\; c_4 > c(n),$$
$$\sigma'_{3,j}(F_\theta) := \{x \in I : \, dist(x, I_j(F,\theta)) < H^{-1+4v/3}\}.$$

From this,

$$\mu(\sigma(F)) \le c_4^{-1} c(n) \mu(\sigma_3(F)) H \Psi(H), \tag{18}$$

where $\sigma_3(F_\theta) = \cup_j \sigma_{3,j}(F_\theta)$. It is clear that $\sigma_{3,j}(F_\theta) \subset \sigma'_{3,j}(F_\theta)$. Moreover, it is easy to see that

$$\sigma_{3,j}(F_\theta) \subset \sigma'_{3,i}(\tilde{F}_\theta) \tag{19}$$

for any $\tilde{F} \in \mathcal{F}_{n,\mathbf{b}_1}(H)$ with $\sigma_{3,i}(\tilde{F}_\theta) \cap \sigma_{3,j}(F_\theta) \ne \emptyset$.

Summing the measures of the essential intervals $\sigma_3(F_\theta)$ gives

$$\sum_{F \in \mathcal{F}_{n,\mathbf{b}_1}(H)} \; \sum_{\substack{j \\ \sigma_{3,j}(F_\theta) \;\; \text{essential}}} \mu(\sigma_{3,j}(F_\theta)) \ll |I|. \tag{20}$$

As $\#\mathbf{b}_1 \ll H^{n-2}$, from (18) and (20), we have

$$\sum_{H=1}^{\infty} \sum_{\mathbf{b}_1} \sum_{F \in \mathcal{F}_{n,\mathbf{b}_1}(H)} \mu(\sigma(F_\theta)) \ll \sum_{H=1}^{\infty} H^{n-1}\Psi(H)|I| < \infty.$$

By the Borel–Cantelli Lemma, the set of those x belonging to infinitely many essential intervals has zero measure.

The proof of the theorem is therefore complete. □

Link to the picture 1 — 1.

References

[1] D. Y. Kleinbock, G. A. Margulis, "Flows on homogeneous spaces and Diophantine approximation on manifolds", *Ann. of Math.*, **148**, (1998), 339–360.

[2] V. V. Beresnevich, S. Velani, "An inhomogeneous transference principle and Diophantine approximation", *Proc. Lond. Math. Soc.*, **101**, (2010), 821–851.

[3] V. I. Bernik, D. Dickinson, M. Dodson, "Approximation of real numbers by values of integer polynomials", *Dokl. Nats. Akad. Nauk Belarusi*, **42**, (1998), 51–54.

[4] V. V. Beresnevich, "A Groshev type theorem for convergence on manifolds", *Acta Math. Hungar.*, **94**, (2002), 99–130.

[5] V. V. Beresnevich, V. I. Bernik, D. Y. Kleinbock, G. A. Margulis, "Metric Diophantine approximation: the Khintchine-Groshev theorem for nondegenerate manifolds", *Mosc. Math. J.*, **2**, (2002), 203–225.

[6] V. I. Bernik, D. Y. Kleinbock, G. A. Margulis, "Khintchine–type theorems on manifolds: the convergence case for standard and multiplicative versions", *Internat. Res. Notices*, **9**, (2001), 453–486.

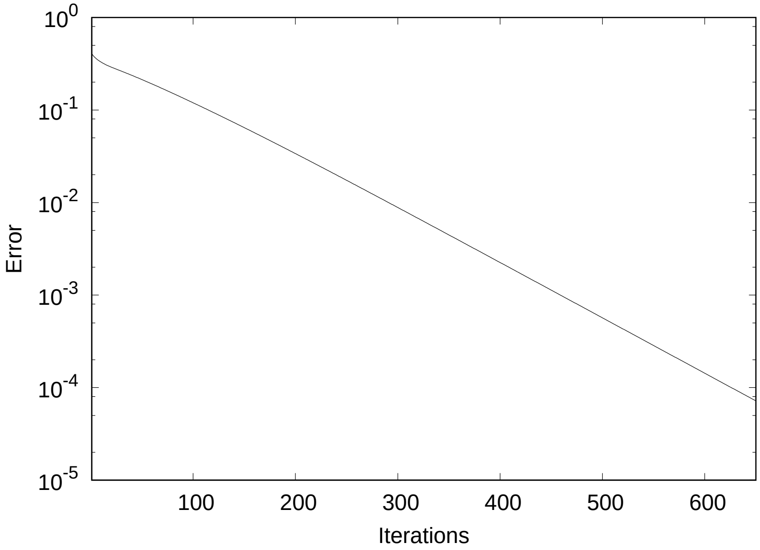

Fig. 1: Example of a picture.

[7] D. Badziahin, V. V. Beresnevich, S. Velani, "Inhomogeneous theory of dual Diophantine approximation on manifolds", *Adv. Math.*, **232**:1, (2013), 1–35.

[8] V. V. Beresnevich, "On a theorem of V. Bernik in the metric theory of Diophantine approximation", *Acta Arith.*, **117**, (2005), 71–80.

[9] N. Budarina, D. Dickinson, "Diophantine approximation on non–degenerate curves with non–monotonic error function", *Bull. Lond. Math. Soc.*, **41**, (2009), 137–146.

[10] V. V. Beresnevich, V. I. Bernik, "On a metrical theorem of W. Schmidt", *Acta Arith.*, **75**:3, (1996), 219–233.

[11] A. Piartly, "Diophantine approximations on submanifolds of Euclidean space", *Funktsional. Anal. i Prilozhen.*, **3**:4, (1969), 59–62.

Received by the editors
August 24, 2013

Н. В. Бударина Неоднородные диофантовы приближения на кривых с немонотонной функцией аппроксимации. *Дальневосточный математический журнал.* 2017. Т. 17. № 1. С. 1–9.

АННОТАЦИЯ

В данной статье доказывается неоднородный аналог теоремы типа Грошева в случае сходимости для невырожденных кривых в евклидовом пространстве, когда функция аппроксимации является не обязательно монотонной. Наш результат естественно включает в себя и обобщает теорему для меры множества точек невырожденных кривых в однородном случае. В доказательстве используются неоднородный метод переноса и метод существенных и несущественных областей Спринджука.

Ключевые слова: *неоднородные диофантовы приближения, теорема Хинчина, невырожденная кривая.*