# Simultaneous distribution of primitive lattice points in convex planar domain

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**Abstract**

Let $\Omega$ denote a compact convex subset of $\mathbf{R}^2$. Suppose that $\Omega$ contains the origin as an inner point. Suppose that $\Omega$ is bounded by the curve $\partial\Omega$, parametrized by $x = r_\Omega(\theta)\cos\theta$, $y = r_\Omega(\theta)\sin\theta$, where the function $r_\Omega$ is continuous and piecewise $C^3$ on $[0, \pi/4]$. For each real $R \geq 1$ we consider the dilation $\Omega_R = \{(Rx, Ry)|(x, y) \in \Omega\}$ of $\Omega$, and the set $\mathcal{F}(\Omega, R)$ of all primitive lattice points inside $\Omega_R$. The purpose of this paper is the study of simultaneous distribution for lengths of segments connecting the origin and primitive lattice points of $\mathcal{F}(\Omega, R)$. For every $\alpha, \beta \in [0, 1]$, consider the set $P(\alpha, \beta, R)$ of fundamental parallelograms for $\mathbf{Z}^2$ of the shape $t_1v+t_2w$ with $t_1, t_2 \in [0, 1]$, defined by points $v = (|v|\cos\theta_v, |v|\sin\theta_v)$, $w = (|w|\cos\theta_w, |w|\sin\theta_w) \in \mathcal{F}(\Omega, R)$, such that $\frac{|v|}{R} \leq \alpha r_\Omega(\theta_v)$ and $\frac{|w|}{R} \leq \beta r_\Omega(\theta_w)$. We establish an asymptotic formula

$$\frac{\#P(\alpha, \beta, R)}{\#\mathcal{F}(\Omega, R)} = 2\int_0^\beta\int_0^\alpha [\alpha' + \beta' \geq 1]d\alpha' d\beta' + O\big(R^{-\frac{1}{3}}\log^{\frac{2}{3}} R\big),$$

where $[\cdot]$ denotes the value of logical expression.

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2010 *Mathematics Subject Classification*: Primary 11L05; Secondary 11H06.

*Key words and phrases*: primitive lattice points, simultaneous distribution.

Author supported by Russian Foundation for Basic Research, Grant 16-51-10018 KO-a.

# 1. Introduction

Let $\Omega$ be a compact convex domain in a plane. Using polar coordinates we write

$$\Omega = \{(r,\varphi)|\ 0 \le r \le r(\varphi) \le 1,\ 0 \le \varphi \le \varphi_0 \le \pi/4\}, \tag{1}$$

where $r = r(\varphi)$ is continuous on $[0, \varphi_0]$. For each real $R \ge 1$ we consider the domain $\Omega_R$ consisting of points $(Rx, Ry)$ with $(x, y) \in \Omega$. Let $\mathcal{F}(\Omega, R)$ denote the set of primitive integer points of $\Omega_R$. We can write $\mathcal{F}(\Omega, R)$ as

$$\mathcal{F}(\Omega,R) = \left\{ A_j \in \Omega_R \cap \mathbf{Z}^2 \;\middle|\; \begin{array}{l} A_j = (x_j, y_j),\ \text{g.c.d.}(x_j, y_j) = 1, \\ \theta_j = \arctan\left(\frac{y_j}{x_j}\right), \\ \theta_{j+1} = \arctan\left(\frac{y_{j+1}}{x_{j+1}}\right), \\ \theta_j < \theta_{j+1},\ 1 \le j < N \end{array} \right\}, \tag{2}$$

where $N$ denotes the cardinality of $\mathcal{F}(\Omega, R)$. We say that the points $A_j$ and $A_{j+1}$ are *consecutive points*, and we say that the rays which have the vertex at $(0, 0)$ and pass through $A_j$ and $A_{j+1}$ respectively are *consecutive rays*.

Boca F. P., Cobeli C., Zaharescu A. have investigated in [1] the distribution of normalized gaps

$$\frac{N}{2\pi}(\theta_2 - \theta_1), \dots, \frac{N}{2\pi}(\theta_N - \theta_{N-1}) \tag{3}$$

between the angles $\theta_1 < \theta_2 < \cdots < \theta_N$. They have obtained an exact formula for this distribution.

A.Ustinov has noted in the paper [2] that the problem of the distribution of values (3) can be easily solved if we know the simultaneous distribution of lengths of segments $d_j, d_{j+1}$ $(1 \le j < N)$, where $d_j = \sqrt{x_j^2 + y_j^2}$. He has established an asymptotic formula for simultaneous distribution of $d_j, d_{j+1}$ $(1 \le j < N)$ when $\Omega$ is a triangle:

**Theorem 1.** *Let* $\Omega$ *be a triangle with vertices* $(0, 0), (1, 0), (1, \tan(\varphi_0))$ *and* $r(\varphi) = 1/\cos(\varphi)$. *Let*

$$\begin{aligned} \Phi(R) = {} & \Phi(R; \varphi_0, \alpha, \beta) = \\ & = \left\{ (A_j, A_{j+1}) \in \mathcal{F}^2(\Omega, R) \;\middle|\; \begin{array}{l} A_j = (x_j, y_j), \\ d_j \le \alpha R r(\theta_j), \\ d_{j+1} \le \beta R r(\theta_{j+1}), \\ \theta_{j+1} \le \varphi_0, \\ 1 \le j < \#\mathcal{F}(\Omega, R) \end{array} \right\}, \end{aligned} \tag{4}$$

$$N_{\varphi_0}(R) = \sum_{j=0}^{\#\mathcal{F}(\Omega,R)-1} [\theta_{j+1} \le \varphi_0]. \tag{5}$$

*Then for any* $\alpha, \beta \in [0,1],\ \varphi_0 \in [0, \pi/4],\ R \geq 2$ *one has*

$$\frac{\#\Phi(R)}{N_{\varphi_0}(R)} = \mathcal{I}(\alpha,\beta) + O\big(R^{-\frac{1}{2}} \log^3 R\big) \text{ as } R \to \infty,$$

*where*

$$\mathcal{I}(\alpha,\beta){=}2\int_0^\beta\int_0^\alpha [\alpha' + \beta' \geq 1] d\alpha' d\beta' = \begin{cases} 0, & \text{if } \alpha+\beta \leq 1, \\ (\alpha+\beta-1)^2, & \text{otherwise.} \end{cases} \quad (6)$$

In the present work we consider a more general situation:

**Theorem 2.** *Let the domain* $\Omega$ *be given by (1). Let* $r(\varphi)$ *be a real function with three continuous derivatives for* $\varphi \in [0, \varphi_0]$. *Suppose that for* $\varphi \in [0, \varphi_0]$ *functions*

$$x(\varphi) = r(\varphi)\cos(\varphi),\ y(\varphi) = r(\varphi)\sin(\varphi), \Psi(\varphi) = x''(\varphi) - 2x'(\varphi)\tan(\varphi)$$

*satisfy the following conditions:*

1. $x'(\varphi) \leq 0, y'(\varphi) \geq 0,\ |x'(\varphi)|, y'(\varphi) < \infty.$
2. *The equation* $\Psi(\varphi) = 0$ *has a finite number of solutions in* $[0, \varphi_0]$.
3. *There is no* $\varphi \in [0, \varphi_0]$ *such that* $\Psi(\varphi) = \Psi'(\varphi) = 0$.

*Then for any* $\alpha, \beta \in [0,1],\ \varphi_0 \in [0, \pi/4]$,

$$\frac{\#\Phi(R)}{N_{\varphi_0}(R)} = \mathcal{I}(\alpha,\beta) + O\big(R^{-\frac{1}{3}} \log^{\frac{2}{3}} R\big) \text{ as } R \to \infty,$$

*where* $\Phi(R), N_{\varphi_0}(R), \mathcal{I}(\alpha,\beta)$ *are given by* $(4) - (6)$.

**Remark 1.** *In particular case when the equation* $\Psi(\varphi) = 0$ *has no solutions in* $[0, \varphi_0]$, *the error term is* $O(R^{-\frac{1}{2}+\varepsilon})$.

In this paper we always assume that the boundary $\partial\Omega$ of $\Omega$ satisfies the conditions of Theorem 2.

## 2. Formula for $\#\Phi(R)$

**Statement 1.** *For any consecutive points* $A_j = (x_j, y_j)$, $A_{j+1} = (x_{j+1}, y_{j+1})$ *of* $\mathcal{F}(\Omega, R)$ *the point* $(x_j + x_{j+1}, y_j + y_{j+1})$ *does not lie in* $\Omega_R$.

*Proof.* Let $A = (x_j + x_{j+1}, y_j + y_{j+1})$ and $A' = (\frac{x_j + x_{j+1}}{d}, \frac{y_j + y_{j+1}}{d})$, where $d = \text{g.c.d.}(x_j + x_{j+1}, y_j + y_{j+1})$. Suppose that $A \in \Omega_R$. Then $A' \in \Omega_R$ and this means that $A' \in \mathcal{F}(\Omega, R)$. We observe that the point $A'$ lies inside the angle generated by consecutive rays, which pass through points $A_j$, $A_{j+1}$. This contradicts (2). $\square$

**Statement 2.** *If $\alpha$ and $\beta$ are non-negative real numbers and $\alpha+\beta<1$, then $\#\Phi(R)=0$.*

*Proof.* Suppose that $\Phi(R)$ is a nonempty set. Then there is a pair $A_j = (x_j, y_j)$, $A_{j+1} = (x_{j+1}, y_{j+1})$ of consecutive elements of $\mathcal{F}(\Omega, R)$ satisfying the relations

$$\begin{aligned} x_j &= \alpha' Rr(\theta_j)\cos(\theta_j) \quad , \quad x_{j+1} = \beta' Rr(\theta_{j+1})\cos(\theta_{j+1}), \\ y_j &= \alpha' Rr(\theta_j)\sin(\theta_j) \quad , \quad y_{j+1} = \beta' Rr(\theta_{j+1})\sin(\theta_{j+1}) \end{aligned}$$

for some $\alpha' \in [0,\alpha]$ and $\beta' \in [0,\beta]$. The condition $\alpha+\beta<1$ leads to the conclusion that the point $A = (x_j + x_{j+1}, y_j + y_{j+1})$ lies below the straight line passing through $A_j$ and $A_{j+1}$. Therefore $A \in \Omega_R$. This contradicts Statement 1. □

**Statement 3.** *For any consecutive points $A_j = (x_j, y_j)$ and $A_{j+1} = (x_{j+1}, y_{j+1})$ of $\mathcal{F}(\Omega, R)$ we have*

$$x_j y_{j+1} - x_{j+1} y_j = \pm 1.$$

*Proof.* We consider the triangle with vertices $(0,0)$, $A_j$, $A_{j+1}$. According to Statement 1 the triangle does not contain elements of the lattice $\mathbf{Z}^2$. So the parallelogram with vertices $(0,0)$, $A_j$, $A_{j+1}$, $(x_j + x_{j+1}, y_j + y_{j+1})$ is a fundamental parallelogram of the lattice $\mathbf{Z}^2$. It is known that the area of this parallelogram is equal to $\mid x_j y_{j+1} - x_{j+1} y_j \mid$ and the determinant of the lattice $\mathbf{Z}^2$ is equal to 1. Hence Statement 3 follows. □

**Lemma 1.** *Let*

$$\mathcal{T}_+(R) = \left\{ (P,P',Q,Q') \;\middle|\; \begin{array}{l} P'Q - PQ' = 1, \\ Q \le Q',\ P \le Q,\ P' \le Q', P' \le Q' \tan(\varphi_0), \\ (Q,P) \in \Omega_{\alpha R},\ (Q',P') \in \Omega_{\beta R},\ (Q+Q', P+P') \notin \Omega_R \end{array} \right\},$$

$$\mathcal{T}_-(R) = \left\{ (P,P',Q,Q') \;\middle|\; \begin{array}{l} P'Q - PQ' = -1, \\ Q \le Q',\ P \le Q,\ P' \le Q', P \le Q \tan(\varphi_0), \\ (Q,P) \in \Omega_{\beta R},\ (Q',P') \in \Omega_{\alpha R},\ (Q+Q', P+P') \notin \Omega_R \end{array} \right\}$$

*be sets of 4-tuples $(P,P',Q,Q') \in \boldsymbol{Z}^4$. Then*

$$\#\Phi(R) = \#\mathcal{T}(R) = \#\mathcal{T}_-(R) + \#\mathcal{T}_+(R),$$

*where $\mathcal{T}(R) = \mathcal{T}_-(R) \bigcup \mathcal{T}_+(R)$.*

*Proof.* It follows from definitions of $\mathcal{T}_-(R)$ and $\mathcal{T}_+(R)$ that $\mathcal{T}_-(R)\bigcap\mathcal{T}_+(R)=\emptyset$.

Let $A_j=(x_j,y_j)$, $A_{j+1}=(x_{j+1},y_{j+1})$ be consecutive points of $\mathcal{F}(\Omega,R)$ and $(A_j,A_{j+1})\in\Phi(R)$. By (1), (2), (4) and Statement 1, Statement 3, setting

$$(P,P',Q,Q')=\begin{cases}(y_j,y_{j+1},x_j,x_{j+1}), & \text{if } x_j\le x_{j+1},\\ (y_{j+1},y_j,x_{j+1},x_j), & \text{if } x_j>x_{j+1},\end{cases}$$

we have $(P,P',Q,Q')\in\mathcal{T}(R)$. Hence $\#\Phi(R)\le\#\mathcal{T}(R)$.

Conversely, putting

$$(y_j,y_{j+1},x_j,x_{j+1})=\begin{cases}(P,P',Q,Q'), & \text{if } (P,P',Q,Q')\in\mathcal{T}_+(R),\\ (P',P,Q',Q), & \text{if } (P,P',Q,Q')\in\mathcal{T}_-(R),\end{cases}$$

we observe that $A_j=(x_j,y_j)$, $A_{j+1}=(x_{j+1},y_{j+1})$ are consecutive points of $\mathcal{F}(\Omega,R)$ and $(A_j,A_{j+1})\in\Phi(R)$. So $\#\Phi(R)\ge\#\mathcal{T}(R)$. The desired conclusion follows. □

Now we are ready to calculate $\#\mathcal{T}_+(R)$. In our context we put $q=Q'$, $u=P'$, $v=Q$. Then Lemma 1 yields the representation

$$\#\mathcal{T}_+(R)=\sum_{q<R}\sum_{u,v=1}^{q}\delta_q(uv-1), \tag{7}$$

where

$$u\le q\tan(\varphi_0),\ (q,u)\in\Omega_{\beta R},\ (vq,uv-1)\in\Omega_{\alpha qR},\ (q(q+v),u(q+v)-1)\notin\Omega_{qR}.$$

Here

$$\delta_q(uv-1)=\begin{cases}1, & \text{if } q|(uv-1),\\ 0, & \text{otherwise}\end{cases}$$

is the indicator function of divisibility by $q$.

The domain $\{(u,v)|(vq,uv-1)\in\Omega_{\alpha qR},\ (q(q+v),u(q+v)-1)\notin\Omega_{qR}\}$ is bounded by curves

$$\{(u,f_1(u))\}=\quad\{(u,v)|v=\alpha Rx(t),\ u=q\tan(t)+\frac{1}{\alpha Rx(t)},\ t\in[0,\varphi_0]\},$$

$$\{(u,f_2(u))\}=\quad\{(u,v)|v=Rx(t)-q,\ u=q\tan(t)+\frac{1}{Rx(t)},\ t\in[0,\varphi_0]\},$$

so (7) may be expressed as

$$\#\mathcal{T}_+(R)=\sum_{q<R}\ \sum_{\substack{u\in(0,q\tan(\varphi_0)]\\(q,u)\in\Omega_{\beta R}}}\ \sum_{f_2(u)<v\le\min\{q,f_1(u)\}}\delta_q(uv-1).$$

We replace the functions $f_1(u), f_2(u)$ by functions $g_1(u,\alpha), g_2(u)$, which we define by

$$\{(u, g_1(u,\alpha))\} = \{(u,v)|v = \alpha Rx(t),\ u = q\tan(t),\ t \in [0,\varphi_0]\}, \tag{8}$$
$$\{(u, g_2(u))\} = \{(u,v)|v = Rx(t) - q,\ u = q\tan(t),\ t \in [0,\varphi_0]\}. \tag{9}$$

This replacing gives the error term $O(1)$. Define

$$S(R,\alpha,\beta) = \sum_{q<R} \sum_{u\in I(q,\beta)} \sum_{g_2(u)<v\le\min\{q,g_1(u,\alpha)\}} \delta_q(uv-1), \tag{10}$$
$$I(q,\beta) = \{u \in (0,q]|(q,u) \in \Omega_{\beta R},\ u \le q\tan(\varphi_0)\}.$$

Then it is clear that

$$\#\mathcal{T}_+(R) = S(R,\alpha,\beta) + O(1). \tag{11}$$

We need the following estimates concerning the number of solutions of congruence $uv \equiv 1(\mod q)$ in the domain $\{(u,v)|u \in (X_1,X_2], v \in (0, f(u)]\}$, obtained by A. Ustinov [3]:

**Lemma 2.** *Let $X_1, X_2, Y$ be a real non-negative numbers, which do not exceed $q$. Then*

$$\sum_{u\in(X_1,X_2]} \sum_{v\in(0,Y]} \delta_q(uv\pm 1) = \frac{Y}{q} \sum_{\substack{u\in(X_1,X_2]\\(q,u)=1}} 1 + O(R_1[q]),$$

*where*

$$R_1[q] \ll \sigma(q)\log^2(q+1)\sqrt{q}.$$

*Here $\sigma(q)$ is the number of divisors of $q$.*

**Lemma 3.** *Let $f(x)$ be a non-negative real function two times differentiable for $[X_1,X_2]$ $(0 \le X_1, X_2 \le q)$, whose derivatives satisfy the condition*

$$\frac{1}{A} \ll |f''(x)| \ll \frac{w}{A}$$

*for some constants $A > 0, w \ge 1$. Then the asymptotic formula*

$$\sum_{u\in(X_1,X_2]} \sum_{0<v\le f(u)} \delta_q(uv\pm 1) = \frac{1}{q} \sum_{\substack{u\in(X_1,X_2]\\ \text{g.c.d.}(q,u)=1}} f(u) + O(R_2[q,A,X_2-X_1]),$$

*is valid. Here*

$$R_2[q,A,X] \ll_w \sigma^{\frac{2}{3}}(q) X A^{-\frac{1}{3}} + X^{\varepsilon}(\sqrt{A} + \sqrt{q}).$$

Now we turn to (10). We write $S(R,\alpha,\beta)$ as

$$S(R,\alpha,\beta) = S_1'(R,\alpha,\beta) + S_1''(R,\alpha,\beta) - S_2(R,\alpha,\beta), \tag{12}$$

where

$$\begin{aligned} S_1'(R,\alpha,\beta) &= \sum_{q\le R}\sum_{u\in I'(q,\alpha,\beta)}\sum_{v\le q}\delta_q(uv-1),\\ S_1''(R,\alpha,\beta) &= \sum_{q\le R}\sum_{u\in I''(q,\alpha,\beta)}\sum_{v\le g_1(u,\alpha)}\delta_q(uv-1),\\ S_2(R,\alpha,\beta) &= \sum_{q\le R}\sum_{u\in I(q,\beta)}\sum_{v\le g_2(u)}\delta_q(uv-1). \end{aligned}$$

Here intervals $I'(q,\alpha,\beta), I''(q,\alpha,\beta)$ are defined by

$$\begin{aligned} I'(q,\alpha,\beta) &= \{u\in I(q,\beta) | g_2(u) < q \le g_1(u,\alpha)\},\\ I''(q,\alpha,\beta) &= \{u\in I(q,\beta) | g_2(u) < g_1(u,\alpha) \le q\}. \end{aligned}$$

According to Lemma 2 and the bound $\sum_{q<R}\sigma(q) \ll R\log R$, we have

$$S_1'(R,\alpha,\beta) = \sum_{q<R}\frac{1}{q}\sum_{\substack{u\in I'(q,\alpha,\beta)\\ \text{g.c.d.}(q,u)=1}} q + O\big(R^{\frac{3}{2}}\log^3 R\big). \tag{13}$$

To estimate two other sums $S_1''(R,\alpha,\beta)$ and $S_2(R,\alpha,\beta)$ we must consider the fact that for fixed natural $q$ the second derivatives of $g_1(u,\alpha)$ and $g_2(u)$ lie within closed intervals containing zero.

**Lemma 4.** *For $S_1''(R,\alpha,\beta)$ and $S_2(R,\alpha,\beta)$ it follows that*

$$\begin{aligned} S_1''(R,\alpha,\beta) &= \sum_{q<R}\frac{1}{q}\sum_{\substack{u\in I''(q,\alpha,\beta)\\ \text{g.c.d.}(q,u)=1}} g_1(u,\alpha) + O\big(R^{2-\frac{1}{3}}\log^{\frac{2}{3}} R\big),\ R\to\infty,\\ S_2(R,\alpha,\beta) &= \sum_{q<R}\frac{1}{q}\sum_{\substack{u\in I(q,\beta)\\ \text{g.c.d.}(q,u)=1}} g_2(u,\alpha) + O\big(R^{2-\frac{1}{3}}\log^{\frac{2}{3}} R\big),\ R\to\infty. \end{aligned}$$

*Proof.* We will prove the lemma for $S_1''(R,\alpha,\beta)$ only as we can easily adapt the proof below for the sum $S_2(R,\alpha,\beta)$. By (8) we conclude that

$$g_1''(u,\alpha) = \frac{\alpha R}{q^2}\cos^4(t)\Psi(t),\ t = \arctan\Big(\frac{u}{q}\Big),$$

where the function $\Psi(t)$ is defined in Theorem 2. This function vanishes at a finite number of points. Without loss of generality we suppose that the equation $\Psi(t)=0$ has only one solution which we denote by $t_0$. We denote the corresponding value of the variable $u$ by $u_0$.

If $t_0 \notin (0,\varphi_0]$, application of Lemma 2 (with $A=\frac{q^2}{R}$) to inner sums over $u,v$ of the sum $S_1''(R,\alpha,\beta)$ gives

$$S_1''(R,\alpha,\beta)=\sum_{q<R}\frac{1}{q}\sum_{\substack{u\in I''(q,\alpha,\beta)\\(q,u)=1}} g_1(u,\alpha)+O\big(R^{\frac{3}{2}+\varepsilon}\big), \tag{14}$$

since

$$\sum_{q<R} R_2\Big[q,\frac{q^2}{R},q\Big] \ll R^{\frac{3}{2}+\varepsilon}.$$

For this case Lemma 4 is proved.

Let $t_0\in(0,\varphi_0]$. Put

$$u_{max}=\max_{u\in I''(q,\alpha,\beta)}\{u\},\ k=[\log_2(u_{max})],$$
$$S(q,J)=\sum_{u\in J\cap I''(q,\alpha,\beta)}\ \sum_{v\le g_1(u,\alpha)}\delta_q(uv-1),$$

where $J$ is the interval.

Let $\Delta\in(0,1)$ be a real number to be specified later. We divide the interval $(0,u_{max}]$ into subintervals $J^{(0)},\ J_i\ (1\le i\le k+1):$

$$J^{(0)}=(u_0-\Delta q,u_0+\Delta q]\bigcap I''(q,\alpha,\beta),$$

$$J_i=\begin{cases}(2^{i-1},2^i]\bigcap I''(q,\alpha,\beta), & \text{if } J^{(0)}=\emptyset,\\ (2^{i-1},2^i]\bigcap I''(q,\alpha,\beta)\backslash(J^{(0)}\bigcap(2^{i-1},2^i]), & \text{otherwise}\end{cases}$$

(some of these intervals may be empty). For the above reasons we write the sum $S_1''(R,\alpha,\beta)$ as

$$S_1''(R,\alpha,\beta)\ =\ \sum_{q<R}\ \sum_{1\le i\le k+1}S(q,J_i)+\sum_{q<R}S(q,J^{(0)}).$$

The set $\{J_i\}_{i=1}^{k+1}$ has subintervals for which intersections with $J^{(0)}$ are non-empty. We denote these ones as $J^{(1)},J^{(2)}$. We apply Lemma 2 to $S(q,J^{(0)})$, replacing $g_1(u,\alpha)$ with the constant $g_1(u_0-\Delta q,\alpha)$. As $\mid g_1'(u,\alpha)\mid\ll\frac{R}{q}$. Then this replacing gives the error term $O(R\Delta^2)$. To other sums we apply Lemma 3 with

$$A=\frac{q^2}{R}\cdot\begin{cases}\Delta^{-1} & \text{— for } J^{(1)},J^{(2)},\\ q\cdot 2^{-i} & \text{— for } J_i,\ \text{not coinciding with } J^{(1)},J^{(2)}.\end{cases}$$

We obtain

$$S_1''(R,\alpha,\beta)=\sum_{q<R}\frac{1}{q}\sum_{\substack{u\in I''(q,\alpha,\beta)\\ \text{g.c.d.}(q,u)=1}} g_1(u,\alpha)+O(R''),$$

where

$$\begin{aligned} R'' &= R_1'' + R_2'', & (15)\\ R_1'' &\ll \sum_{q<R} R_1[q] + \sum_{q<R} R\Delta^2, & (16)\\ R_2'' &\ll \sum_{q<R}\sum_{i<\log q} R_2\left[q, \frac{q^3}{R\cdot 2^i}, 2^i\right] + \sum_{q<R} R_2\left[q, \frac{q^2}{R\Delta}, \Delta\right]. & (17)\end{aligned}$$

The sums on the right of (16) may be estimated by $R^{\frac{3}{2}}\log^3 R$ and $R^2\Delta^2$ respectively. Using Lemma 3 we represent the sum in the right hand side of (17) as a sum of three terms $\Sigma_1$, $\Sigma_2$, $\Sigma_3$ :

$$\begin{aligned} \Sigma_1 &= \sum_{q<R}\sum_{j<\log q} \sigma^{\frac{2}{3}}(q)2^j\left(\frac{R\cdot 2^j}{q^3}\right)^{\frac{1}{3}} + \sum_{q<R}\sigma^{\frac{2}{3}}(q)\Delta\left(\frac{R\Delta}{q^2}\right)^{\frac{1}{3}},\\ \Sigma_2 &= \sum_{q<R}\sum_{j<\log q} 2^{\varepsilon j}\sqrt{\frac{q^3}{R\cdot 2^j}} + \sum_{q<R}\Delta^{\varepsilon}\sqrt{\frac{q^2}{R\Delta}},\\ \Sigma_3 &= \sum_{q<R}\sum_{j<\log q} 2^{\varepsilon j}\sqrt{q} + \sum_{q<R}\Delta^{\varepsilon}\sqrt{q}.\end{aligned}$$

In $\Sigma_1$ we see that the first term dominates, so we may omit the second term. Therefore

$$\begin{aligned}\Sigma_1 &\ll R^{\frac{1}{3}}\sum_{q<R}\sigma^{\frac{2}{3}}(q)q^{-1}\sum_{j<\log q}2^{\frac{4}{3}j} \ll R^{\frac{1}{3}}\sum_{q<R}\sigma^{\frac{2}{3}}(q)q^{\frac{1}{3}} \ll R^{\frac{1}{3}}\left(\sum_{q<R}\sigma(q)\right)^{\frac{2}{3}}\left(\sum_{q<R}q\right)^{\frac{1}{3}} \ll\\ &\ll R^{1+\frac{2}{3}}\log^{\frac{2}{3}} R.\end{aligned}$$

Also in $\Sigma_2$ the second term may be omitted and in the first term the sum over $q, j$ is restricted to pairs with $\frac{q^3}{R\cdot 2^j} \geq 1$. In all intervals $J_j$ $(1 \leq j \leq k+1)$ we have $g''(u) \gg \frac{R\cdot\Delta}{q^2}$, then $\frac{R\cdot 2^j}{q^3} \gg \frac{R\cdot\Delta}{q^2}$. So

$$\Sigma_2 \ll \sum_{q<R}\sum_{j<\log q} 2^{\varepsilon j}\sqrt{\frac{q^3}{R\cdot 2^j}} \ll \sum_{q<R}\sum_{j<\log q} 2^{\varepsilon j}\sqrt{\frac{q^2}{R\cdot\Delta}} \ll R^{-\frac{1}{2}}\Delta^{-\frac{1}{2}}\sum_{q<R} q^{1+\varepsilon} \ll R^{\frac{3}{2}+\varepsilon}\Delta^{-\frac{1}{2}}.$$

As $\Sigma_3 \ll R^{\frac{3}{2}+\varepsilon}$ we have $R_2'' \ll R^{1+\frac{2}{3}}\log^{\frac{2}{3}} R + R^{\frac{3}{2}+\varepsilon}\Delta^{-\frac{1}{2}}$. From (15) we get an estimate of the error term for $S_1''(R,\alpha,\beta)$ :

$$R'' \ll R^{\frac{3}{2}}\log^3 R + R^2\Delta^2 + R^{1+\frac{2}{3}}\log^{\frac{2}{3}} R + R^{\frac{3}{2}+\varepsilon}\Delta^{-\frac{1}{2}}.$$

Now we have to choose the parameter $\Delta$ in such a way that $R^2\Delta^2 \asymp R^{\frac{3}{2}+\varepsilon}\Delta^{-\frac{1}{2}}$. Then we get $\Delta = R^{-\frac{1-2\varepsilon}{5}}$. This gives the result of Lemma 4 for $S_1''(R,\alpha,\beta)$. □

Let $F(u,q,\alpha)$ denote the function $F(u,q,\alpha)=\min\{q,g_1(u,\alpha)\}-g_2(u)$. The relations (12), (13) and Lemma (4) give

$$S(R,\alpha,\beta)=\sum_{q<R}\frac{1}{q}\sum_{\substack{u\in I(q,\beta)\\ \text{g.c.d.}(q,u)=1}}F(u,q,\alpha)+O\big(R^{2-\frac{1}{3}}\log^{\frac{2}{3}}R\big). \qquad (18)$$

By (8) and (9) we have

$$\frac{1}{q}\sum_{\substack{u\in I(q,\beta)\\ \text{g.c.d.}(q,u)=1}}F(u,q,\alpha)=\frac{1}{q}\sum_{\delta|q}\mu(\delta)\sum_{\substack{u\in I(q,\beta)\\ \delta|u}}F(u,q,\alpha).$$

From the identity

$$\sum_{\substack{u\in I(q,\beta)\\ \delta|u}}F(u,q,\alpha)=\frac{1}{\delta}\int_0^q[u\in I(q,\beta)]F(u,q,\alpha)du+O(q)$$

and relations (8), (9) we have

$$\int_0^q[u\in I(q,\beta)]F(u,q,\alpha)du=q^2\int_0^1\int_0^1\left[t\le t_q,t\le\varphi_0,\frac{R}{q}x(t)-1<v\le\alpha\frac{R}{q}x(t)\right]dvd\tan(t),$$

where the value $t_q$ is given by $q=\beta Rx(t_q)$. Now the main term in (18), which we denote as $S^*(R,\alpha,\beta)$, can be written in the form

$$S^*(R,\alpha,\beta)=\sum_{\delta<R}\mu(\delta)S'\left(\frac{R}{\delta}\right), \qquad (19)$$

where

$$S'(R)=\sum_{q<R}q\int_0^1\int_0^1\left[t\le t_q,t\le\varphi_0,\frac{R}{q}x(t)-1<v\le\alpha\frac{R}{q}x(t)\right]dvd\tan(t).$$

Here we take into account that the remainder $\sum_{q<R}\frac{1}{q}\sum_{\delta|q}q\ll R\log R$ is less than the error term in (18).

To evaluate $S'(R)$ we change the order of the summation and the integration, then we replace the sum with the integral, taking into account that the error term is of order $R$. Thus we have

$$S'(R)=R^2\frac{1}{2}\int_0^{\varphi_0}x^2(t)d\tan(t)\int_0^1\left[\frac{1}{\beta}-1<v<\frac{\alpha}{1-\alpha}\right]\left(\min\left\{\frac{\alpha^2}{v^2},\beta^2\right\}-\frac{1}{(v+1)^2}\right)dv+$$
$$+O(R).$$

Applying Statement 2, we obtain $S'(R)=R^2S_\Omega\cdot I(\alpha,\beta)+O(R)$, where $I(\alpha,\beta)$ is defined by the formula

$$I(\alpha,\beta)=[\alpha+\beta\ge1]\cdot[\beta\ge1/2]\cdot\begin{cases}(\alpha+\beta-1)^2, & \text{if } \alpha\le1/2,\\ 2(\beta-1/2)^2-(\alpha-\beta)^2, & \text{if } 1/2<\alpha\le\beta,\\ 2(\beta-1/2)^2, & \text{if } \alpha>\beta\end{cases}$$

and $S_\Omega$ denotes the area of the domain $\Omega$. Combining the above result with (10),(11), (18), (19) we get an asymptotic formula for $\#\mathcal{T}_+(R)$ :

$$\#\mathcal{T}_+(R) = \frac{R^2}{\zeta(2)} S_\Omega \cdot I(\alpha,\beta) + O\big(R^{2-\frac{1}{3}} \log^{\frac{2}{3}} R\big).$$

To prove the asymptotic formula for $\#\mathcal{T}_-(R)$, we proceed similarly to (7). We deduce

$$\#\mathcal{T}_-(R) = \sum_{q<R} \sum_{u,v=1}^{q} \delta_q(uv+1),$$

where

$$u \le q\tan(\varphi_0) - 1/v,\ (q,u) \in \Omega_{\alpha R},\ (vq, uv-1) \in \Omega_{\beta q R},\ (q(q+v), u(q+v)-1) \notin \Omega_{qR}.$$

According to (8)-(11) we have $\mathcal{T}_-(R) = S(R,\beta,\alpha) + O(R)$. Then

$$\#\mathcal{T}_-(R) = \frac{R^2}{\zeta(2)} S_\Omega \cdot I(\beta,\alpha) + O\big(R^{2-\frac{1}{3}} \log^{\frac{2}{3}} R\big).$$

At last we note, that $I(\alpha,\beta) + I(\beta,\alpha) = \mathcal{I}(\alpha,\beta)$; and by Lemma 1 for $\#\Phi(R)$ we obtain the asymptotics

$$\#\Phi(R) = \frac{R^2}{\zeta(2)} S_\Omega \cdot \mathcal{I}(\alpha,\beta) + O\big(R^{2-\frac{1}{3}} \log^{\frac{2}{3}} R\big). \tag{20}$$

## 3. Proof of Theorem 2

Theorem 2 follows from (20) and the asymptotic formula for $\#\mathcal{F}(\Omega,R)$ :

$$\begin{aligned}\#\mathcal{F}(\Omega,R) &= \sum_{\substack{(x,y)\in F(\Omega,R)\\ \text{g.c.d.}(x,y)=1}} 1 = \sum_{(x,y)\in F(\Omega,R)} \sum_{\delta|\text{g.c.d.}(x,y)} \mu(\delta) = \sum_{\delta<R} \mu(\delta) \sum_{(x,y)\in F(\Omega,R/\delta)} 1 = \\ &= R^2 \cdot S_\Omega \sum_{\delta<R} \frac{\mu(\delta)}{\delta^2} + O(R\log(R)) = \frac{R^2}{\zeta(2)} \cdot S_\Omega + O(R\log(R)).\end{aligned}$$

## 4. Acknowledgements

I am grateful to N.G.Moshchevitin for his helpful comments and useful remarks.

## References

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