Kloosterman Sums and Continued Fractions

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General idea

Let \( f(x) = \prod_{n \geq 1} \left( 1 - \frac{x}{n} \right)^{\frac{1}{n}} \). Reduced regular continued fraction

\[
\frac{a_0}{1 + \frac{a_1}{1 + \frac{a_2}{1 + \ddots}}}
\]

where \( a_0, a_1, a_2, \ldots \) - done sequence \( (a_j, j) \).

Theorem 8 (see [6])

1. Let \( f(x) = \prod_{n \geq 1} \left( 1 - \frac{x}{n} \right)^{\frac{1}{n}} \) and \( \gamma = \lim_{x \to \infty} \frac{f(x)}{x} \). Then

\[
\gamma = \frac{1}{\zeta(2)} = \frac{1}{\pi^2}.
\]

Moreover, if \( b / a \geq 3 \), then

\[
\zeta(2) \approx \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}.
\]

Fast Euclidean algorithms

There are three main Euclidean algorithms: standard correct and odd. They are based essentially on standard division

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Gauss — Kuz’min statistics

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Reduced bases in two-dimensional lattices

Reduced bases are important in different number theory algorithms (e.g., joint optimization on elliptic curves). The form of reduced bases is a generalization of a reduced basis in a two-dimensional lattice. Theorem 8 (see [6])

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Conclusions

This approach continues existing observations, methods from geometry of numbers, and analytic number theory. It gives an effective tool for studying continued fractions and lattice point problems. Many applications can be found in the following articles.


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