Voronoi – Minkowski 3-D continued fractions

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September 3, 2013
Let $S$ be a subset of $\mathbb{R}^2_{\geq 0}$. Consider the boundary of the set

$$S \oplus \mathbb{R}^2_{\geq 0} = \{ s + r \mid s \in S, r \in \mathbb{R}^2_{\geq 0} \}.$$

In other words, this broken line is the boundary of the union of copies of the positive quadrant shifted by vertices of the set $S$. 
Let $S$ be a subset of $\mathbb{R}_{\geq 0}^2$. Consider the boundary of the set

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In other words, this broken line is the boundary of the union of copies of the positive quadrant shifted by vertices of the set $S$. 
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Let $S$ be a subset of $\mathbb{R}_3^{\geq 0}$. The *Voronoi-Minkowski polyhedron* for $S$ is the boundary of the set

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We assume that

1. $S$ has no accumulation points;
2. $S$ is in general position: each plane parallel to a coordinate plane contains at most one point of $S$. 

Voronoi-Minkowski complex
For a nonempty point set $T \subset \mathbb{R}^s$ $\text{Box}(T)$ is the least possible parallelepiped circumscribed about $T$.

More formally: if

$$|T|_i = \max\{|x_i| : x = (x_1, \ldots, x_s) \in T\} \quad (i = 1, \ldots, s),$$

then

$$\text{Box}(T) = [-|T|_1, |T|_1] \times \ldots \times [-|T|_s, |T|_s].$$
A point $\gamma$ in a lattice $\Gamma$ is called a *relative (local) minimum* of the lattice $\Gamma$ in the sense of Voronoi (or simply a *minimum*) if the Box($\gamma$) is *free* (it contains no points of the lattice $\Gamma$ different from its vertices and the origin).

2-D example:
The Box($\gamma_1, \gamma_2$) is called *extreme* if it is *free* and if, at the same time, it has on each of its faces at least one lattice point. In other words it is impossible to extend this parallelepiped in any coordinate direction so that the resulting parallelepiped still contains no nonzero lattice points.
When we consider local minima or extreme parallelepipeds signs are not important for us. We can remove them.
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Instead of lattice \( \Gamma \) we can consider a set \( |\Gamma| \subset \mathbb{R}^s \) where

\[
|\Gamma| = \{(|x|, |y|, |z|) : (x, y, z) \in \Gamma\}.
\]
As it was proved by Voronoi, we can consider a classical continued fraction as a sequence of local minima (halls) or extreme parallelepipeds (hills).
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Local minima and Klein polyhedron: (in 2-D case)

local minima=vertices of Klein polyhedron
In 3-D case vertices of Klein polyhedron are always local minima, but converse is not true (Bykovski, 2006). In other words local minima have more rich structure (they can lie on the faces of Klein polyhedron.)
The Box($\gamma_1, \gamma_2, \gamma_3$) is called *extreme* if it is *free* (it contains no lattice points other than the origin) and if, at the same time, it has on each of its faces at least one lattice point.
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A set of vectors (s.t. $v_i \neq v_j$) $S$ in the lattice $\Gamma$ is said to be *minimal* if the Box($S$) contains no points of $\Gamma$ except the origin. In particular, a minimal system of order 1 is a local minimum, minimal systems of order 3 gives extreme parallelepiped.
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If $\{\gamma_1, \gamma_2\}$ is a minimal system of order 2 then $\gamma_1$ and $\gamma_2$ are *neighbours*. 
Minkowski graph
Voronoi (=Minkowski*) graph

Here coordinates of vertices in space and on the plane $x + y + z = 0$ are concordant

Alexey Ustinov (Khabarovsk) Voronoi – Minkowski 3-D continued fractions
Some reasons

Why do this objects are interesting and important?

- Good algorithms.
- Periodicity for algebraic numbers.
- “Vahlen’s theorem”.
- “Gauss measure”.
- Possibility to apply “hard” (analytical) methods based on Kloosterman sums.
Minkowski and Voronoi proposed algorithms for finding fundamental units in cubic fields. (All pictures above correspond to the case of totally real cubic fields. In the case of complex cubic fields parallelepiped must be replaced by cylinders.)
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Voronoi considered chains of local minima.

They were able to do all calculations by hand 😊
Theorem (Lagrange’s Continued Fraction Theorem.)

The real roots of quadratic expressions with integral coefficients have periodic continued fractions.
Some reasons

Periodicity

**Theorem (Lagrange’s Continued Fraction Theorem.)**

The real roots of quadratic expressions with integral coefficients have periodic continued fractions.

Two main examples (the beginning of *Markov spectrum*) are

\[
\frac{1 + \sqrt{5}}{2} = 2 \cos \frac{2\pi}{5} = [1; 1, \ldots, 1, \ldots] = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ldots}}},
\]

\[
\sqrt{2} = 2 \cos \frac{2\pi}{8} = [1; 2, \ldots, 2, \ldots] = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \ldots}}},
\]
Some reasons
Periodicity

With quadratic irrational \( \alpha \) we can associate a lattice \( \Gamma(\alpha) \) with basis \((1, 1)\) and \((\alpha, \beta)\) where \( \beta \) is conjugate of \( \alpha \) (second root of the same quadratic equation.)
Periodical continued fraction of \( \alpha \) describes periodical structure of local minima of \( \Gamma(\alpha) \).
With quadratic irrational $\alpha$ we can associate a lattice $\Gamma(\alpha)$ with basis $(1, 1)$ and $(\alpha, \beta)$ where $\beta$ is conjugate of $\alpha$ (second root of the same quadratic equation.)

Periodical continued fraction of $\alpha$ describes periodical structure of local minima of $\Gamma(\alpha)$.

With cubic irrationality $\alpha$ (from totally real cubic field) we can associate 3-D lattice with basis $(1, 1, 1)$, $(\alpha, \beta, \gamma)$, $(\alpha^2, \beta^2, \gamma^2)$, where $\beta$ and $\gamma$ are conjugates of $\alpha$. 
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Voronoi–Minkowski graph for such lattices is doubly periodic (totally real cubic field has 2 fundamental units).
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Two mains examples arise from cubic numbers \( \alpha = 2 \cos \frac{2\pi}{7} \) and \( \alpha = 2 \cos \frac{2\pi}{9} \) (associated with first two extremal Davenport cubic forms).
Some reasons

Periodicity

The Voronoi graph for $\alpha = 2 \cos \frac{2\pi}{7}$

\[
\begin{align*}
\alpha^2 - \alpha - 1 & \\
\alpha^2 & \\
-\alpha - \alpha^2 & \\
\alpha & \\
1 - \alpha^2 & \\
1 & \\
-1 - \alpha & \\
\alpha^2 - 2 & \\
\alpha^2 + \alpha - 2 & \\
-\alpha^2 - \alpha + 1 & \\
\end{align*}
\]
The Voronoi graph for $\alpha = 2 \cos \frac{2\pi}{9}$
Some reasons
Vahlen’s theorem

Denote by $\frac{p_n}{q_n} = [a_0; a_1, \ldots, a_n]$ convergents to a given number $\alpha = [a_0; a_1, \ldots, a_n, \ldots]$.

Vahlen’s theorem: for $p/q = p_{n-1}/q_{n-1}$ or $p/q = p_n/q_n$

$$\left| \alpha - \frac{p}{q} \right| \leq \frac{1}{2q^2}$$

can be translated to the lattice language. The equivalent statement:

$\gamma_a = (a_1, a_2), \gamma_b = (b_1, b_2)$ is a minimal system on lattice $\Gamma$, then

$$\min\{|a_1 a_2|, |b_1 b_2|\} \leq \frac{1}{2} \det \Gamma.$$
Some reasons

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Vahlen’s theorem has a stronger form:

$$|a_1 a_2| + |b_1 b_2| \leq \det \Gamma.$$
Some reasons
3-D Vahlen’s theorem

Theorem (Avdeeva and Bykovskii, 2006)

If

\( \gamma_a = (a_1, a_2, a_3), \quad \gamma_b = (b_1, b_2, b_3), \quad \gamma_c = (c_1, c_2, c_3), \)

is a minimal system on lattice \( \Gamma \), then

\[
| a_1 a_2 a_3 | + | b_1 b_2 b_3 | + | c_1 c_2 c_3 | \leq \det \Gamma.
\]
Theorem (Avdeeva and Bykovskii, 2006)

If

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\[ |a_1 a_2 a_3| + |b_1 b_2 b_3| + |c_1 c_2 c_3| \leq \det \Gamma. \]

This theorem can be regarded as a sharpening of the estimate

\[ |a_1 a_2 a_3| + |b_1 b_2 b_3| + |c_1 c_2 c_3| \leq 3 \det \Gamma, \]

which follows from Minkowski's convex body theorem.
Some reasons
Gauss measure

In 2-D case minimal couple \( \gamma_a = (a_1, a_2) \), \( \gamma_b = (b_1, b_2) \) is always a basis of a given lattice (Voronoï):
Some reasons

Gauss measure

We can associate with minimal system \( \gamma_a = (a_1, a_2) \), \( \gamma_b = (b_1, b_2) \) the matrix \( \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \) with diagonal dominance: \( |a_1| > |b_1|, |b_2| > |a_2| \).
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- permutation of rows or columns (renumbering of the vectors or of the coordinates axes);
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- permutation of rows or columns (renumbering of the vectors or of the coordinates axes);
- changing the signs of all elements in a column (changing the direction of a coordinate axis);
- multiplication of a row by a nonzero number (rescaling one of the coordinate axes, possibly in combination with changing the orientation of this axis).
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\[
\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & x \\ -y & 1 \end{pmatrix} \text{ where } 0 < x < 1, 0 < y < 1. \quad \text{Box}(\gamma_a, \gamma_b) \rightarrow [-1, 1]^2.
\]
Some reasons
Gauss measure

Gaussian measure

\[ d\mu = \frac{dx
dy}{(1 + xy)^2} = \frac{dx\,dy}{\begin{vmatrix} 1 & x \\ y & 1 \end{vmatrix}^2} \]

defined for \((x, y) \in [0, 1]^2\) describes typical behavior of classical continued fractions. From geometrical point of view this density function describes distribution of vectors from bases \(\begin{pmatrix} 1 & x \\ -y & 1 \end{pmatrix}\) on the sides of unit square.
Some reasons
Gauss measure

In 2-D case minimal couple $\gamma_a = (a_1, a_2), \gamma_b = (b_1, b_2)$ is always a basis of a given lattice and $(a_1 b_1 \ b_1) \sim (1 \ x) \ y \ 1$ where $0 < x < 1, 0 < y < 1$. 
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3-D surprise (Minkowski): either minimal triple $\gamma_a = (a_1, a_2, a_3)$, $\gamma_b = (b_1, b_2, b_3)$, $\gamma_c = (c_1, c_2, c_3)$ is a basis and corresponding matrix equivalent to

$$
\begin{pmatrix}
1 & x_2 & \pm x_3 \\
-y_2 & 1 & y_3 \\
z_1 & -z_2 & 1
\end{pmatrix}
$$
In 2-D case minimal couple $\gamma_a = (a_1, a_2)$, $\gamma_b = (b_1, b_2)$ is always a basis of a given lattice and $(a_1' \ b_1') \sim \begin{pmatrix} 1 \\ -y \\ 1 \end{pmatrix}$ where $0 < x < 1$, $0 < y < 1$.

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\begin{pmatrix}
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-y_2 & 1 & y_3 \\
z_1 & -z_2 & 1
\end{pmatrix}
$$

or it is degenerate ($\det(\gamma_a, \gamma_b, \gamma_c) = 0$) and for some combination of signs

$$
\gamma_a \pm \gamma_b \pm \gamma_c = 0.
$$
Some reasons

Gauss measure

In 2-D case minimal couple $\gamma_a = (a_1, a_2)$, $\gamma_b = (b_1, b_2)$ is always a basis of a given lattice and $(a_1 b_1) \sim (1 x)$ where $0 < x < 1$, $0 < y < 1$.

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$$\gamma_a \pm \gamma_b \pm \gamma_c = 0.$$ 

For bases $(x_2, x_3, y_1, y_3, z_1, z_2) \in \text{subset of } [0, 1]^6$ defined by some simple liner inequalities depending on the sign before $x_3$. 
The 3-D analogue of Gaussian measure

$$d\mu = \frac{dx_2 \, dx_3 \ldots \, dz_2}{\begin{vmatrix} 1 & x_2 & \pm x_3 \\ -y_2 & 1 & y_3 \\ z_1 & -z_2 & 1 \end{vmatrix}^3}$$

describes distribution of basis vector on some subset of $[0, 1]^6$. 
The 3-D analogue of Gaussian measure

\[ d\mu = \frac{dx_2 \, dx_3 \ldots \, dz_2}{\left| \begin{array}{ccc} 1 & x_2 & \pm x_3 \\ -y_2 & 1 & y_3 \\ z_1 & -z_2 & 1 \end{array} \right|^3} \]

describes distribution of basis vector on some subset of \([0, 1]^6\).

The same measure describes behavior of Klein polyhedrons. The difference is in measure space. Measure space varies for different types of 3-D continued fractions.
Some reasons
Analytical tool: Kloosterman sums

In 2-D problems we study $2 \times 2$ matrices with fixed determinant:

$$\det \begin{pmatrix} a & b \\ -c & d \end{pmatrix} = N,$$

where $a_2 \leq b_2, b_1 \leq a_1$.
Some reasons
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We can fix $a$ and consider a congruence

$$bc \equiv N \pmod{a}.$$
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For each solution $(b, c)$ a pair $(zb, z^{-1}c)$ where $zz^{-1} \equiv 1 \pmod{a}$ is also a solution.
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Propagating a solution we have 3 degrees of freedom $(u, t, z \in \mathbb{Z})$:

$$\begin{vmatrix} a & b \\ -c & d \end{vmatrix} = N \quad \Rightarrow \quad \begin{vmatrix} a & zb + ua \\ -z^{-1}c + ta & * \end{vmatrix} = N$$
Some reasons
Analytical tool: Kloosterman sums

In such a situation

$$\begin{vmatrix}
  a & zb + ua \\
- z^{-1}c + ta & *
\end{vmatrix} = N$$

we can average over $z$ and apply estimations of Kloosterman sums

$$K_a(m, n) = \sum_{\substack{z=1 \\
(a,z)=1}}^{a} e^{2\pi i \frac{mz+nz^{-1}}{a}}.$$
Some reasons
Analytical tool: Kloosterman sums and 3-D → 2-D reduction

Let \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), \( q = \det A \neq 0 \) and

\[
\begin{vmatrix} A & x_1 \\ x_3 & x_2 \end{vmatrix} = N. \\
\begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = N.
\]

Propagating a solution we have 5 degrees of freedom \( (u, v, s, t, z \in \mathbb{Z}) \):
Some reasons
Analytical tool: Kloosterman sums and 3-D $\rightarrow$ 2-D reduction

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $q = \det A \neq 0$ and

$$\begin{vmatrix} A & x_1 \\ x_2 & x_3 \\ x_4 & x_5 \end{vmatrix} = N.$$ 

Propagating a solution we have 5 degrees of freedom ($u, v, s, t, z \in \mathbb{Z}$):

$$\begin{vmatrix} A & z^{-1}x_1 + ua + vb \\ z^{-1}x_2 + uc + vd & zx_3 + sa + tc \\ zx_4 + sb + td & * \end{vmatrix} = P,$$

where $zz^{-1} \equiv 1 \pmod{q}$. 

[Note: The asterisk (*) indicates an unspecified element.]
Some reasons
Analytical tool: Kloosterman sums and 3-D $\rightarrow$ 2-D reduction

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $q = \det A \neq 0$ and

$$\begin{vmatrix} A & x_1 \\ x_2 & x_3 \\ x_4 & x_5 \end{vmatrix} = N.$$  

Propagating a solution we have 5 degrees of freedom ($u$, $v$, $s$, $t$, $z \in \mathbb{Z}$):

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where $zz^{-1} \equiv 1 \pmod{q}$.
Averaging over $z$ we can apply Kloosterman sums again.
Some reasons
Analytical tool: Kloosterman sums and 3-D → 2-D reduction

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\begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = N.
\]
\[
\begin{vmatrix} x_3 & x_4 & x_5 \\ a & b & c \\ d \end{vmatrix} = N.
\]

Propagating a solution we have 5 degrees of freedom (\( u, v, s, t, z \in \mathbb{Z} \)):
\[
\begin{vmatrix} z^{-1}x_1 + ua + vb \\ z^{-1}x_2 + uc + vd \\ zx_3 + sa + tc \\ zx_4 + sb + td \end{vmatrix} = P,
\]
where \( zz^{-1} \equiv 1 \pmod{q} \).

Averaging over \( z \) we can apply Kloosterman sums again.
Linnik and Skubenko (1964) used this argument studying distribution of points on a variety defined by equation \( \det(x_{ij}) = N \) (\( i, j = 1, 2, 3 \)).
Open problems

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Geometry:
- characterize possible Minkowski (Voronoi) graphs arising from 3-D lattices;
- characterize possible periodic Minkowski (Voronoi) graphs arising from 3-D algebraic lattices;
- is it always possible to draw infinite Voronoi graph without accumulation points, keeping its geometry and using edges of 3 directions?
- Are there any connections with singularity resolutions in toric geometry?
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Geometry:
- characterize possible Minkowski (Voronoï) graphs arising from 3-D lattices;
- characterize possible PERIODIC Minkowski (Voronoï) graphs arising from 3-D ALGEBRAIC lattices;
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Thank you for your attention!