The Farey Fraction Spin Chain and Gauss—Kuz'min Statistics for Quadratic Irrationals

Alexey Ustinov

Institute of Applied Mathematics (Khabarovsk)
Russian Academy of Sciences (Far Eastern Branch)

June 28, 2012

The Farey Fraction Spin Chain and Gauss—Kuz'min Statistics for Quadratic Irrationals

Alexey Ustinov

Institute of Applied Mathematics (Khabarovsk)
Russian Academy of Sciences (Far Eastern Branch)

June 28, 2012

The Farey Fraction Spin Chain

Consider the following simple model of a magnet: a chain composed of some molecules (spins), each of which can point either up $\uparrow$ or down $\downarrow$

$$\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\uparrow\uparrow=\uparrow^3\downarrow^2\uparrow^4 .$$

In statistical physics we need to define the probability of a given configuration. Usually it depends on the energy E and the temperature T:

$$p = \frac{e^{-E/T}}{Z},$$

where Z is just the normalizing factor.
There are different ways to assign an energy to each state of a spin chain.

The Farey Fraction Spin Chain

Consider the following simple model of a magnet: a chain composed of some molecules (spins), each of which can point either up $\uparrow$ or down $\downarrow$

$$\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\uparrow\uparrow=\uparrow^3\downarrow^2\uparrow^4 .$$

In statistical physics we need to define the probability of a given configuration. Usually it depends on the energy E and the temperature T:

$$p = \frac{e^{-E/T}}{Z},$$

where Z is just the normalizing factor.
There are different ways to assign an energy to each state of a spin chain. .

The Farey Fraction Spin Chain

Kleban and Özlük (1999) introduced Farey Fraction Spin Chain model based on the products of matrices

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In this model $\uparrow = A$, $\downarrow = B$. For example

$$\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\uparrow\uparrow = \uparrow^3\downarrow^2\uparrow^4 = A^3B^2A^4.$$

For a given configuration they proposed to assign the energy

$$E(\uparrow^{a_1}\downarrow^{a_2}\uparrow^{a_3}\ldots) = \log\left(\mathrm{Tr}\left(A^{a_1}B^{a_2}A^{a_3}\ldots\right)\right).$$

In particular

$$E(A^n) = \log 2, \qquad E((AB)^n) \asymp n \quad \text{(typical)}.$$

The Farey Fraction Spin Chain

Kleban and Özlük (1999) introduced Farey Fraction Spin Chain model based on the products of matrices

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In this model $\uparrow = A$, $\downarrow = B$. For example

$$\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\uparrow\uparrow = \uparrow^3\downarrow^2\uparrow^4 = A^3B^2A^4.$$

For a given configuration they proposed to assign the energy

$$E(\uparrow^{a_1}\downarrow^{a_2}\uparrow^{a_3}\ldots) = \log\left(\mathrm{Tr}\left(A^{a_1}B^{a_2}A^{a_3}\ldots\right)\right).$$

In particular

$$E(A^n) = \log 2, \qquad E((AB)^n) \asymp n \quad \text{(typical)}.$$

Several authors (Kallies, Özlük, Peter, Snyder, Boca, Fiala) considered the following problem: determine the number of states with energy bounded by N. Let

$$\Psi(N) = \left|\left\{C \in \langle A, B\rangle : 3 \leqslant \mathrm{Tr}\, C \leqslant N\right\}\right|.$$

Kallies, Özlük, Peter, Snyder (2001):

$$\Psi(N) = \frac{N^2 \log N}{\zeta(2)} + O(N^2 \log\log N).$$

Several authors (Kallies, Özlük, Peter, Snyder, Boca, Fiala) considered the following problem: determine the number of states with energy bounded by N. Let

$$\Psi(N) = \left|\left\{C \in \langle A, B\rangle : 3 \leqslant \mathrm{Tr}\, C \leqslant N\right\}\right|.$$

Kallies, Özlük, Peter, Snyder (2001):

$$\Psi(N) = \frac{N^2 \log N}{\zeta(2)} + O(N^2 \log\log N).$$

Boca (2007):

$$\Psi(N) = N^2(c_1 \log N + c_0) + O_\varepsilon\left(N^{7/4+\varepsilon}\right),$$

where

$$c_1 = \frac{1}{\zeta(2)}, \quad c_2 = \frac{1}{\zeta(2)}\left(\gamma - \frac{3}{2} - \frac{\zeta'(2)}{\zeta(2)}\right).$$

Theorem (AU, 2012)

$$\Psi(N) = N^2(c_1 \log N + c_0) + O_\varepsilon\left(N^{3/2+\varepsilon}\right).$$

This result follows from Weil's (+ Estermann) bound

$$|K_q(m,n)| \leqslant \sigma_0(q) \cdot (m,n,q)^{1/2} \cdot q^{1/2}.$$

for Kloosterman sums

$$K_q(m,n) = \sum_{\substack{x,y=1 \\ xy\equiv 1 \pmod q}}^{q} e^{2\pi i \frac{mx+ny}{q}}.$$

Boca (2007):

$$\Psi(N) = N^2(c_1 \log N + c_0) + O_\varepsilon\left(N^{7/4+\varepsilon}\right),$$

where

$$c_1 = \frac{1}{\zeta(2)}, \quad c_2 = \frac{1}{\zeta(2)}\left(\gamma - \frac{3}{2} - \frac{\zeta'(2)}{\zeta(2)}\right).$$

Theorem (AU, 2012)

$$\Psi(N) = N^2(c_1 \log N + c_0) + O_\varepsilon\left(N^{3/2+\varepsilon}\right).$$

This result follows from Weil's (+ Estermann) bound

$$|K_q(m,n)| \leqslant \sigma_0(q) \cdot (m,n,q)^{1/2} \cdot q^{1/2}.$$

for Kloosterman sums

$$K_q(m,n) = \sum_{\substack{x,y=1 \\ xy\equiv 1 \pmod q}}^{q} e^{2\pi i \frac{mx+ny}{q}}.$$

Boca (2007):

$$\Psi(N) = N^2(c_1 \log N + c_0) + O_\varepsilon\left(N^{7/4+\varepsilon}\right),$$

where

$$c_1 = \frac{1}{\zeta(2)}, \quad c_2 = \frac{1}{\zeta(2)}\left(\gamma - \frac{3}{2} - \frac{\zeta'(2)}{\zeta(2)}\right).$$

Theorem (AU, 2012)

$$\Psi(N) = N^2(c_1 \log N + c_0) + O_\varepsilon\left(N^{3/2+\varepsilon}\right).$$

This result follows from Weil's (+ Estermann) bound

$$|K_q(m,n)| \leqslant \sigma_0(q) \cdot (m,n,q)^{1/2} \cdot q^{1/2}.$$

for Kloosterman sums

$$K_q(m,n) = \sum_{\substack{x,y=1 \\ xy\equiv 1 \pmod q}}^{q} e^{2\pi i \frac{mx+ny}{q}}.$$

It is better to split a problem in two parts

$$\Psi(N) = 2\big(\Psi_{ev}(N) + \Psi_{odd}(N)\big)$$

and consider even and odd spin chains separately:

$$\Psi_{ev}(N) = \big|\{C = A^{a_1}B^{a_2}\dots B^{a_{2n}} : 3 \leqslant \operatorname{Tr} C \leqslant N\}\big|$$
$$\Psi_{odd}(N) = \big|\{C = A^{a_1}B^{a_2}\dots A^{a_{2n+1}} : 3 \leqslant \operatorname{Tr} C \leqslant N\}\big|.$$

Asymptotic formula for $\Psi(N)$ follows from

$$\Psi_{ev}(N) = \frac{\log 2}{2\zeta(2)}N^2 + O(N^{3/2+\varepsilon})$$
$$\Psi_{odd}(N) = \frac{N^2}{2\zeta(2)}\left(\log N - \log 2 + \gamma - \frac{3}{2} - \frac{\zeta'(2)}{\zeta(2)}\right) + O(N^{3/2+\varepsilon}).$$

It is better to split a problem in two parts

$$\Psi(N) = 2\big(\Psi_{ev}(N) + \Psi_{odd}(N)\big)$$

and consider even and odd spin chains separately:

$$\Psi_{ev}(N) = \big|\{C = A^{a_1}B^{a_2}\dots B^{a_{2n}} : 3 \leqslant \mathrm{Tr}\, C \leqslant N\}\big|$$
$$\Psi_{odd}(N) = \big|\{C = A^{a_1}B^{a_2}\dots A^{a_{2n+1}} : 3 \leqslant \mathrm{Tr}\, C \leqslant N\}\big|.$$

Asymptotic formula for $\Psi(N)$ follows from

$$\Psi_{ev}(N) = \frac{\log 2}{2\zeta(2)}N^2 + O(N^{3/2+\varepsilon})$$
$$\Psi_{odd}(N) = \frac{N^2}{2\zeta(2)}\left(\log N - \log 2 + \gamma - \frac{3}{2} - \frac{\zeta'(2)}{\zeta(2)}\right) + O(N^{3/2+\varepsilon}).$$

Let

$$\frac{a}{b} = [a_0; a_1, \dots, a_s + \dots] = a_0 + \cfrac{1}{a_1 + \ddots + \cfrac{1}{a_s + \dots}},$$

be standard continued fraction expansion with $a_0 \in \mathbb{Z}$, $a_1, \dots, a_s$, $\dots \in \mathbb{N}$.

If ω is a quadratic number, its **conjugate** will be denoted by ω^*.

A quadratic number $\omega \in (0, 1)$ is said to be **reduced** if its continued fraction expansion is such that $\omega = [0; \overline{a_1, \dots, a_n}]$.
Let $\mathcal{R}$ be the set of all reduced quadratic numbers and $\rho(\omega)$ is the length of ω (the length of corresponding closed geodesics of upper half-plane).

Let

$$\frac{a}{b} = [a_0; a_1, \ldots, a_s + \ldots] = a_0 + \cfrac{1}{a_1 + \ddots + \cfrac{1}{a_s + \ldots}},$$

be standard continued fraction expansion with $a_0 \in \mathbb{Z}$, $a_1, \ldots, a_s, \ldots \in \mathbb{N}$.

If ω is a quadratic number, its **conjugate** will be denoted by ω^*.

A quadratic number $\omega \in (0, 1)$ is said to be **reduced** if its continued fraction expansion is such that $\omega = [0; \overline{a_1, \ldots, a_n}]$.
Let $\mathcal{R}$ be the set of all reduced quadratic numbers and $\rho(\omega)$ is the length of ω (the length of corresponding closed geodesics of upper half-plane).

Let

$$\frac{a}{b} = [a_0; a_1, \dots, a_s + \dots] = a_0 + \cfrac{1}{a_1 + \ddots + \cfrac{1}{a_s + \dots}},$$

be standard continued fraction expansion with $a_0 \in \mathbb{Z}$, $a_1, \dots, a_s, \dots \in \mathbb{N}$.

If ω is a quadratic number, its **conjugate** will be denoted by ω^*.

A quadratic number $\omega \in (0, 1)$ is said to be **reduced** if its continued fraction expansion is such that $\omega = [0; \overline{a_1, \dots, a_n}]$.
Let $\mathcal{R}$ be the set of all reduced quadratic numbers and $\rho(\omega)$ is the length of ω (the length of corresponding closed geodesics of upper half-plane).

Faivre, 1992:

$$\sum_{\rho(\omega)<x} 1 \sim \frac{e^x \log 2}{2\zeta(2)}$$

It is the special type of prime geodesic theorems studied by Linnik, Skubenko, Margulis, Sarnak, Duke, Pollicott...

Boca's result

$$\Psi_{ev}(N) = \frac{\log 2}{2\zeta(2)} N^2 + O(N^{7/4+\varepsilon})$$

equivalent to

$$\sum_{\rho(\omega)<x} 1 = \frac{e^x \log 2}{2\zeta(2)} + O_\varepsilon\left(e^{(\frac{7}{8}+\varepsilon)x}\right).$$

Better error term $O(N^{3/2+\varepsilon})$ in asymptotic formula for $\Psi_{ev}(N)$ gives better error term $O\left(e^{(\frac{3}{4}+\varepsilon)x}\right)$ in last formula.

We need Gauss — Kuz'min statistics to look inside spin chains.

Faivre, 1992:

$$\sum_{\rho(\omega)<x} 1 \sim \frac{e^x \log 2}{2\zeta(2)}$$

It is the special type of prime geodesic theorems studied by Linnik, Skubenko, Margulis, Sarnak, Duke, Pollicott...

Boca's result

$$\Psi_{ev}(N) = \frac{\log 2}{2\zeta(2)} N^2 + O(N^{7/4+\varepsilon})$$

equivalent to

$$\sum_{\rho(\omega)<x} 1 = \frac{e^x \log 2}{2\zeta(2)} + O_\varepsilon\left(e^{(\frac{7}{8}+\varepsilon)x}\right).$$

Better error term $O(N^{3/2+\varepsilon})$ in asymptotic formula for $\Psi_{ev}(N)$ gives better error term $O\left(e^{(\frac{3}{4}+\varepsilon)x}\right)$ in last formula.

We need Gauss — Kuz'min statistics to look inside spin chains.

Faivre, 1992:

$$\sum_{\rho(\omega)<x} 1 \sim \frac{e^x \log 2}{2\zeta(2)}$$

It is the special type of prime geodesic theorems studied by Linnik, Skubenko, Margulis, Sarnak, Duke, Pollicott...
Boca's result

$$\Psi_{ev}(N) = \frac{\log 2}{2\zeta(2)} N^2 + O(N^{7/4+\varepsilon})$$

equivalent to

$$\sum_{\rho(\omega)<x} 1 = \frac{e^x \log 2}{2\zeta(2)} + O_\varepsilon\left(e^{(\frac{7}{8}+\varepsilon)x}\right).$$

Better error term $O(N^{3/2+\varepsilon})$ in asymptotic formula for $\Psi_{ev}(N)$ gives better error term $O\left(e^{(\frac{3}{4}+\varepsilon)x}\right)$ in last formula.
We need Gauss — Kuz'min statistics to look inside spin chains.

Gauss —Kuz'min statistics

Let $\alpha \in (0,1)$, $\alpha = [0; a_1, a_2, \ldots, a_n, \ldots]$.

Theorem (Gauss — Kuz'min, 1800 + 1928)

$$\operatorname{mes}\{\alpha \in (0,1) : [0; a_n, a_{n+1}, \ldots] \leqslant x\} \to \log_2(1+x) = \frac{1}{\log 2} \int_0^x \frac{dt}{1+t}$$

This theorem has following generalization:

$$\operatorname{mes}\{\alpha \in (0,1) : [0; a_n, a_{n+1}, \ldots] \leqslant x, [0; a_{n-1}, \ldots, a_2, a_1] \leqslant y\} \to$$
$$\to \log_2(1+xy) = \frac{1}{\log 2} \int_0^x \int_0^x \frac{dt_1\, dt_2}{(1+t_1 t_2)^2} \qquad (n \to \infty).$$

Conjecture (Arnold, 1993)

Rational numbers and quadratic irrationals satisfy Gauss — Kuz'min law.

Gauss — Kuz'min statistics for rational numbers were studied by Avdeeva — Bykovskii (2002–2004).

Conjecture (Arnold, 1993)

Rational numbers and quadratic irrationals satisfy Gauss — Kuz'min law.

Gauss — Kuz'min statistics for rational numbers were studied by Avdeeva — Bykovskii (2002–2004).

Theorem (A.U., 2005)

For any region Ω with "good" boundary

$$\frac{1}{R^2 \mathrm{Vol}(\Omega)} \sum_{(a/R, b/R) \in \Omega} s_x(a/b) = \frac{2\log(x+1)}{\zeta(2)} \log R + C_\Omega(x) + O(R^{-1/5+\varepsilon}).$$

Conjecture (Arnold, 1993)

Rational numbers and quadratic irrationals satisfy Gauss — Kuz'min law.

Gauss — Kuz'min statistics for rational numbers were studied by Avdeeva — Bykovskii (2002–2004).

Theorem (A.U., 2005)

For any region Ω with "good" boundary

$$\frac{1}{R^2 \mathrm{Vol}(\Omega)} \sum_{(a/R, b/R)\in\Omega} s_x(a/b) = \frac{2\log(x+1)}{\zeta(2)} \log R + C_\Omega(x) + O(R^{-1/5+\varepsilon}).$$

The same law holds for even spin chains and reduced quadratic irrationals (after suitable definitions).

Continued fractions and 2×2 matrices

Let $\mathcal{M}$ be the set of integer matrices

$$S = \begin{pmatrix} p & p' \\ q & q' \end{pmatrix} = \begin{pmatrix} p(S) & p'(S) \\ q(S) & q'(S) \end{pmatrix}$$

such that $\det S = \pm 1$, and

$$1 \leqslant q \leqslant q', \quad 0 \leqslant p \leqslant q, \quad 1 \leqslant p' \leqslant q'.$$

The following map

$$[0; a_1, a_2, \ldots, a_n] \mapsto \begin{pmatrix} p & p' \\ q & q' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix}.$$

is one-to-one correspondence between rationales $\alpha \in (0, 1)$ and matrices $S \in \mathcal{M}$.

Continued fractions and 2×2 matrices

Let $\mathcal{M}$ be the set of integer matrices

$$S = \begin{pmatrix} p & p' \\ q & q' \end{pmatrix} = \begin{pmatrix} p(S) & p'(S) \\ q(S) & q'(S) \end{pmatrix}$$

such that $\det S = \pm 1$, and

$$1 \leqslant q \leqslant q', \quad 0 \leqslant p \leqslant q, \quad 1 \leqslant p' \leqslant q'.$$

The following map

$$[0; a_1, a_2, \ldots, a_n] \mapsto \begin{pmatrix} p & p' \\ q & q' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \ldots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix}.$$

is one-to-one correspondence between rationales $\alpha \in (0, 1)$ and matrices $S \in \mathcal{M}$.

Continued fractions and 2×2 matrices

If

$$\begin{pmatrix} p & p' \\ q & q' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix}.$$

then partial quotients $a - 1, \ldots, a_n$ can be reconstructed by expansions

$$\frac{p}{q} = [0; a_1, \ldots, a_{n-1}], \qquad \frac{p'}{q'} = [0; a_1, \ldots, a_n],$$

or

$$\frac{p}{p'} = [0; a_n, \ldots, a_2], \qquad \frac{q}{q'} = [0; a_n, \ldots, a_1].$$

$\mathcal{M} = \mathcal{M}_+ \sqcup \mathcal{M}_-$ (depending on sign of determinant).

Even spin chains

Equivalent definition of number of spin chains with bounded energy:

$$\Psi_{ev}(N) = \left| \left\{ S = \begin{pmatrix} p & p' \\ q & q' \end{pmatrix} \in \mathcal{M}_+ : \mathrm{Tr}(S) = p + q' \leqslant N \right\} \right| .$$

Gauss — Kuz'min statistics for even spin chains are counted by the function

$$\Psi_{ev}(x, y; N) = \left| \left\{ \begin{pmatrix} p & p' \\ q & q' \end{pmatrix} \in \mathcal{M}_+ : \frac{p'}{q'} \leqslant x, \frac{q}{q'} \leqslant y, p + q' \leqslant N \right\} \right| .$$

Even spin chains

Equivalent definition of number of spin chains with bounded energy:

$$\Psi_{ev}(N) = \left|\left\{ S = \begin{pmatrix} p & p' \\ q & q' \end{pmatrix} \in \mathcal{M}_+ : \mathrm{Tr}(S) = p + q' \leqslant N \right\}\right|.$$

Gauss — Kuz'min statistics for even spin chains are counted by the function

$$\Psi_{ev}(x, y; N) = \left|\left\{ \begin{pmatrix} p & p' \\ q & q' \end{pmatrix} \in \mathcal{M}_+ : \frac{p'}{q'} \leqslant x, \frac{q}{q'} \leqslant y, p + q' \leqslant N \right\}\right|.$$

Even spin chains

Equivalent definition of number of spin chains with bounded energy:

$$\Psi_{ev}(N) = \left| \left\{ S = \begin{pmatrix} p & p' \\ q & q' \end{pmatrix} \in \mathcal{M}_+ : \mathrm{Tr}(S) = p + q' \leqslant N \right\} \right| .$$

Gauss — Kuz'min statistics for even spin chains are counted by the function

$$\Psi_{ev}(x, y; N) = \left| \left\{ \begin{pmatrix} p & p' \\ q & q' \end{pmatrix} \in \mathcal{M}_+ : \ \frac{p'}{q'} \leqslant x, \frac{q}{q'} \leqslant y, p + q' \leqslant N \right\} \right| .$$

Theorem (AU, 2012)

$$\Psi_{ev}(x, y; N) = \frac{\log(1 + xy)}{2\zeta(2)} N^2 + O(N^{3/2+\varepsilon}).$$

Odd spin chains

Theorem (AU, 2012)

$$\Psi_{odd}(x,y;N) = \frac{N^2}{2\zeta(2)}\left(\log N + \log\frac{xy}{x+y} + \gamma - \frac{3}{2} - \frac{\zeta'(2)}{\zeta(2)}\right) + \\ + O(N^{3/2+\varepsilon}) + O\left(\frac{x+y}{xy}N^{1+\varepsilon}\right).$$

Main term here is constructed from matrices $\begin{pmatrix} p & p' \\ q & q' \end{pmatrix}$ with $p' = o(q')$, $q = o(q')$ and has nothing common with Gauss — Kuz'min statistics.

The main tool

Lemma

Let $q \geqslant 1$, $0 \leqslant P_1, P_2 \leqslant q$. *Then*

$$\sum_{0<x\leqslant P_2} \sum_{\substack{0<y\leqslant P_2 \\ xy\equiv 1 \pmod q}} 1 = \frac{\varphi(q)}{q^2} P_1 P_2 + O(q^{1/2+\varepsilon}).$$

The main tool

Lemma

Let $q \geqslant 1$, $0 \leqslant P_1, P_2 \leqslant q$. *Then*

$$\sum_{0<x\leqslant P_2} \sum_{\substack{0<y\leqslant P_2 \\ xy\equiv 1 \pmod q}} 1 = \frac{\varphi(q)}{q^2} P_1 P_2 + O(q^{1/2+\varepsilon}).$$

Lemma

Let $q \geqslant 1$, $0 \leqslant P_1, P_2 \leqslant q$, $f(x) = a \pm x$ *is a linear function such that* $0 \leqslant f(P_1),\ f(P_2) \leqslant q$. *Then*

$$\sum_{P_1<x\leqslant P_2} \sum_{\substack{0<y\leqslant f(x) \\ xy\equiv 1 \pmod q}} 1 = \frac{\varphi(q)}{q^2} \int_{P_1}^{P_2} f(x)dx + O(q^{1/2+\varepsilon}).$$

The length of a quadratic irrational

Let $\mathbb{H} = \{(x, y); y > 0\}$ be the hyperbolic plane with its classical complete metric

$$ds^2 = y^{-2}(dx^2 + dy^2).$$

For this metric the curvature of $\mathbb{H}$ is constant and equal to -1.

Elements of $PSL(2, \mathbb{R})$ are isometries of $\mathbb{H}$.

Let $M = \mathbb{H}/PSL(2, \mathbb{Z})$ be the modular surface.

The length of a quadratic irrational

The geodesics $\gamma : \mathbb{R} \to \mathbb{H}$ for the hyperbolic metric are supported by vertical half-lines and the half-circles centered on the real axis. The geodesics of M are by definition the $p \circ \gamma$ where $\gamma : \mathbb{R} \to \mathbb{H}$ is a geodesic of $\mathbb{H}$ and $p : \mathbb{H} \to M$ the canonical projection.

The following theorem is known and gives the closed geodesics (i.e. periodic geodesics) on the modular surface.

The length of a quadratic irrational

The geodesics $\gamma : \mathbb{R} \to \mathbb{H}$ for the hyperbolic metric are supported by vertical half-lines and the half-circles centered on the real axis. The geodesics of M are by definition the $p \circ \gamma$ where $\gamma : \mathbb{R} \to \mathbb{H}$ is a geodesic of $\mathbb{H}$ and $p : \mathbb{H} \to M$ the canonical projection.

The following theorem is known and gives the closed geodesics (i.e. periodic geodesics) on the modular surface.

The length of a quadratic irrational

The geodesics $\gamma : \mathbb{R} \to \mathbb{H}$ for the hyperbolic metric are supported by vertical half-lines and the half-circles centered on the real axis. The geodesics of M are by definition the $p \circ \gamma$ where $\gamma : \mathbb{R} \to \mathbb{H}$ is a geodesic of $\mathbb{H}$ and $p : \mathbb{H} \to M$ the canonical projection.

The following theorem is known and gives the closed geodesics (i.e. periodic geodesics) on the modular surface.

Theorem

(i) Let γ be a geodesic of $\mathbb{H}$ joining a quadratic number ω and its conjugate ω^*. Then $p \circ \gamma$ is a closed geodesic of M and all the closed geodesics on M arise in this way.
(ii) The length of $p \circ \gamma$ is given by $\rho(\omega) = 2 \log \varepsilon_0(\omega)$.

closed geodesics

Theorem

(i) Let γ be a geodesic of $\mathbb{H}$ joining a quadratic number ω and its conjugate ω^*. Then $p \circ \gamma$ is a closed geodesic of M and all the closed geodesics on M arise in this way.
(ii) The length of $p \circ \gamma$ is given by $\rho(\omega) = 2\log\varepsilon_0(\omega)$.

Let $AX^2 + BX + C \in \mathbb{Z}[X]$ be the minimal equation of quadratic irrational ω in $\mathbb{Z}$ $(A > 0, (A, B, C) = 1)$ and $\Delta = B^2 - 4AC$. Then $\varepsilon_0(\omega) = \frac{1}{2}(x_0 + \sqrt{\Delta}y_0)$ is the fundamental solution of the Pell equation

$$X^2 - \Delta Y^2 = 4.$$

In the field $\mathbb{Q}(\sqrt{\Delta})$ number ω has *trace* $\mathrm{tr}(\omega) = \omega + \omega^* = -B/A$ and *norm* $\mathcal{N}(\omega) = \omega\omega^* = C/A$.

Theorem

(i) Let γ be a geodesic of $\mathbb{H}$ joining a quadratic number ω and its conjugate ω^*. Then $p \circ \gamma$ is a closed geodesic of M and all the closed geodesics on M arise in this way.
(ii) The length of $p \circ \gamma$ is given by $\rho(\omega) = 2 \log \varepsilon_0(\omega)$.

Let $AX^2 + BX + C \in \mathbb{Z}[X]$ be the minimal equation of quadratic irrational ω in $\mathbb{Z}$ $(A > 0, (A, B, C) = 1)$ and $\varDelta = B^2 - 4AC$. Then $\varepsilon_0(\omega) = \frac{1}{2}(x_0 + \sqrt{\Delta} y_0)$ is the fundamental solution of the Pell equation

$$X^2 - \Delta Y^2 = 4.$$

In the field $\mathbb{Q}(\sqrt{\Delta})$ number ω has *trace* $\mathrm{tr}(\omega) = \omega + \omega^* = -B/A$ and *norm* $\mathcal{N}(\omega) = \omega\omega^* = C/A$.

Let ω be reduced and

$$\omega = [0; \overline{a_1, a_2, \dots, a_n}]$$

with period $n = \mathrm{per}(\omega)$. Due to Galois theorem

$$-1/\omega^* = [0; \overline{a_n, \dots, a_1}].$$

For reduced $\omega = [0; \overline{a_1, \dots, a_n}]$ denote by

$$\mathrm{per}_e(\omega) = \begin{cases} n, & \text{if } n = \mathrm{per}(\omega) \text{ is even;} \\ 2n, & \text{if } n = \mathrm{per}(\omega) \text{ is odd} \end{cases}$$

even period of ω.

Fundamental unit ε_0 is the same for numbers

$$\begin{aligned}\omega_1 &= [0; \overline{a_1, a_2, \ldots, a_n}],\\ \omega_2 &= [0; \overline{a_2, a_3, \ldots, a_1}],\\ \omega_3 &= [0; \overline{a_3, a_4, \ldots, a_2}], \ldots\end{aligned}$$

because (Smith's formula)

$$\varepsilon_0^{-1}(\omega) = \omega_1\omega_2 \ldots \omega_{\mathrm{per}_e(\omega)-1}.$$

Fundamental unit ε_0 is the same for numbers

$$\begin{aligned}\omega_1 &= [0; \overline{a_1, a_2, \ldots, a_n}],\\ \omega_2 &= [0; \overline{a_2, a_3, \ldots, a_1}],\\ \omega_3 &= [0; \overline{a_3, a_4, \ldots, a_2}], \ldots\end{aligned}$$

because (Smith's formula)

$$\varepsilon_0^{-1}(\omega) = \omega_1\omega_2 \ldots \omega_{\mathrm{per}_e(\omega)-1}.$$

If A is any statement that can be true or false, then we'll use the following bracketed notation

$$[A] = \begin{cases} 1, & \text{if A is true;}\\ 0, & \text{else}\end{cases}$$

The following sum counts Gauss — Kuz'min statistics for reduced quadratic irrationals

$$r(x,y;N)=\sum_{\substack{\omega\in\mathcal{R}\\ \varepsilon_0(\omega)\leqslant N}}[\omega\leqslant x,\ -1/\omega^*\leqslant y].$$

Numbers $\omega_1,\omega_2,\ldots,\omega_n$ presented in the sum $r(x,y;N)$ simultaneously.

$$r(x,y;N)=$$

$$=\sum_{\substack{\omega\in\mathcal{R}\\ \varepsilon_0(\omega)\leqslant N}}\frac{1}{\mathrm{per}_e(\omega)}\sum_{j=1}^{\mathrm{per}_e(\omega)}\Big[[0;a_{j+1},a_{j+2},\ldots]\leqslant x,\ [0;a_j,a_{j-1},\ldots]\leqslant y\Big].$$

It means that for number ω we count Gauss — Kuz'min statistics for each place in the period. From geometrical point of view we study local behavior of closed geodesics.

The following sum counts Gauss — Kuz'min statistics for reduced quadratic irrationals

$$r(x,y;N) = \sum_{\substack{\omega\in\mathcal{R}\\ \varepsilon_0(\omega)\leqslant N}} [\omega \leqslant x,\ -1/\omega^* \leqslant y].$$

Numbers $\omega_1, \omega_2, \ldots, \omega_n$ presented in the sum $r(x,y;N)$ simultaneously.

$$r(x,y;N) =$$

$$= \sum_{\substack{\omega\in\mathcal{R}\\ \varepsilon_0(\omega)\leqslant N}} \frac{1}{\mathrm{per}_e(\omega)} \sum_{j=1}^{\mathrm{per}_e(\omega)} \Big[[0;a_{j+1},a_{j+2},\ldots] \leqslant x,\ [0;a_j,a_{j-1},\ldots] \leqslant y\Big].$$

It means that for number ω we count Gauss — Kuz'min statistics for each place in the period. From geometrical point of view we study local behavior of closed geodesics.

Theorem

Let $0 \leqslant x, y \leqslant 1$ *and* $N \geqslant 2$. *Then*

$$r(x,y;N) = \Psi_{ev}(x,y;N) + O(N^{3/2+\varepsilon}) = \\ = \frac{\log(1+xy)}{2\zeta(2)} N^2 + O(N^{3/2+\varepsilon}).$$

We can construct the map from even spin chains to the set of reduced quadratic irrationals:

$$B^{a_1}A^{a_2}\dots A^{a_{2m}} \mapsto \omega = [0; \overline{a_1, \dots, a_{2m}}].$$

It is not one-to-one correspondence:

$$B^2A^3B^2A^3 \mapsto \quad \omega = [0; \overline{2,3,2,3}] = [0; \overline{2,3}],$$
$$B^2A^3 \mapsto \quad \omega = [0; \overline{2,3}].$$

But exceptions are rare (continued fraction is at least twice shorter) and fall inside error term.

Theorem

Let $0 \leqslant x, y \leqslant 1$ *and* $N \geqslant 2$. *Then*

$$r(x, y; N) = \Psi_{ev}(x, y; N) + O(N^{3/2+\varepsilon}) = \\ = \frac{\log(1 + xy)}{2\zeta(2)} N^2 + O(N^{3/2+\varepsilon}).$$

We can construct the map from even spin chains to the set of reduced quadratic irrationals:

$$B^{a_1} A^{a_2} \ldots A^{a_{2m}} \mapsto \omega = [0; \overline{a_1, \ldots, a_{2m}}].$$

It is not one-to-one correspondence:

$$B^2 A^3 B^2 A^3 \mapsto \quad \omega = [0; \overline{2, 3, 2, 3}] = [0; \overline{2, 3}],$$
$$B^2 A^3 \mapsto \quad \omega = [0; \overline{2, 3}].$$

But exceptions are rare (continued fraction is at least twice shorter) and fall inside error term.

Main reasons

Traces for spin chains and for quadratic irrationals are concordant: if $\omega = [0; \overline{a_1, \ldots, a_{2m}}]$, $l = \mathrm{per}_e(\omega)$, $2m = kl$, then

$$\mathrm{Tr}(B^{a_1} A^{a_2} \ldots A^{a_{2m}}) = \mathrm{tr}(\varepsilon_0^k(\omega)).$$

This map preserves trace and Gauss — Kuz'min statistics:

$$\omega = [0; \overline{a_1, \ldots, a_{2m}}] \approx [0; a_1, \ldots, a_{2m}],$$
$$-1/\omega^* = [0; \overline{a_{2m}, \ldots, a_1}] \approx [0; a_{2m}, \ldots, a_1].$$

Main reasons

Traces for spin chains and for quadratic irrationals are concordant:
if $\omega = [0; \overline{a_1, \ldots, a_{2m}}]$, $l = \mathrm{per}_e(\omega)$, $2m = kl$, then

$$\mathrm{Tr}(B^{a_1} A^{a_2} \ldots A^{a_{2m}}) = \mathrm{tr}(\varepsilon_0^k(\omega)).$$

This map preserves trace and Gauss — Kuz'min statistics:

$$\omega = [0; \overline{a_1, \ldots, a_{2m}}] \approx [0; a_1, \ldots, a_{2m}],$$
$$-1/\omega^* = [0; \overline{a_{2m}, \ldots, a_1}] \approx [0; a_{2m}, \ldots, a_1].$$

Main reasons

Then traces for spin chains and for quadratic irrationals are concordant:
if $\omega = [0; \overline{a_1, \dots, a_{2m}}]$, $l = \mathrm{per}_e(\omega)$, $2m = kl$, then

$$\mathrm{Tr}(B^{a_1} A^{a_2} \dots A^{a_{2m}}) = \mathrm{tr}(\varepsilon_0^k(\omega)).$$

This map preserves trace and Gauss — Kuz'min statistics:

$$\omega = [0; \overline{a_1, \dots, a_{2m}}] \approx [0; a_1, \dots, a_{2m}],$$
$$-1/\omega^* = [0; \overline{a_{2m}, \dots, a_1}] \approx [0; a_{2m}, \dots, a_1].$$

Main reasons

For $k \geqslant 1$

$$0 < \operatorname{tr}(\varepsilon_0^k(\omega)) - \varepsilon_0^k(\omega) < 1/2$$

(ε_0 is Pisot number).
In the simplest case ($x = y = 1$)

$$\Psi_{ev}(x, y; N) = \sum_{k=1}^{\infty} \sum_{\substack{\omega \in \mathcal{R} \\ \operatorname{tr}(\varepsilon_0^k(\omega)) \leqslant N}} 1 \approx \sum_{\substack{\omega \in \mathcal{R} \\ \operatorname{tr}(\varepsilon_0(\omega)) \leqslant N}} 1 \approx \sum_{\substack{\omega \in \mathcal{R} \\ \varepsilon_0(\omega) \leqslant N}} 1 = r(x, y; N)$$

Thank you for your attention!