On statistical properties of 3D Voronoi-Minkowski continued fractions

Alexey Ustinov

Institute of Applied Mathematics (Khabarovsk)
Russian Academy of Sciences (Far Eastern Branch)

July 3, 2014

Georgy Voronoï (1868–1908) and Hermann Minkowski (1864-1909) looked pretty similar and shared nearly parallel biographies (including their untimely death);
they met once at the ICM in Heidelberg 1904. They founded Geometry of Numbers – a new branch of mathematics, around 1895.

MINKOWSKI H. Generalisation de la theorie des fractions continues. — *Ann. de l'Ecole Norm.*, 1896, 13, 41–60.

VORONOÏ G. F. *On a Generalization of the Algorithm of Continued Fractions (Doctoral Dissertation)*. — Warsaw, 1896. (195 pp. in reprinted edition)

Two translations of Voronoï's thesis are available:

MINKOWSKI H. Generalisation de la theorie des fractions continues. — *Ann. de l'Ecole Norm.*, 1896, 13, 41–60.

VORONOÏ G. F. *On a Generalization of the Algorithm of Continued Fractions (Doctoral Dissertation)*. — Warsaw, 1896. (195 pp. in reprinted edition)

Two translations of Voronoï's thesis are available:

DELONE, B. N. FADDEEV D. K. Theory of irrationalities of third degree, — *Travaux Inst. Math. Stekloff*, 11, Acad. Sci. USSR, MoscowLeningrad, 1940.

VORONOÏ G. F. *On a Generalization of the Algorithm of Continued Fractions (Doctoral Dissertation)* — unofficial translation by Emma Lehmer (exists as pdf-document).

“The present work was completely finished and the printing begun when there was received in Warsaw No. 2 13th Vol. of *Annales Scientifique de l’École Normale Supérieure*. In this no. is the article by H. Minkowski ‘Généralisation de la théorie des fractions continues’…

…

G. F. Voronoï, Warsaw 24th May 1896.”

Previous 3D generalizations of continued fraction algorithm were considered by

- Euler (???),
- Jacobi (1868),
- Hermite (1845),
- Poincaré (1885),
- Hurwitz (1894),
- Klein (1895).

Kloosterman sums

Solutions (x, y) of the congruence

$$xy \equiv 1 \pmod{a}$$

are uniformly distributed in the square $[1, a] \times [1, a]$.

Kloosterman sums

Solutions (x, y) of the congruence

$$xy \equiv 1 \pmod{a}$$

are uniformly distributed in the square $[1, a] \times [1, a]$.
This fact follows from non-trivial bounds for Kloosterman sums

$$K_a(m, n) = \sum_{\substack{x,y=1 \\ xy \equiv 1 \pmod{a}}}^{a} e^{2\pi i \frac{mx+ny}{a}}.$$

Kloosterman sums

Solutions (x, y) of the congruence

$$xy \equiv 1 \pmod{a}$$

are uniformly distributed in the square $[1, a] \times [1, a]$.
This fact follows from non-trivial bounds for Kloosterman sums

$$K_a(m, n) = \sum_{\substack{x,y=1 \\ xy \equiv 1 \pmod{a}}}^{a} e^{2\pi i \frac{mx+ny}{a}}.$$

Solutions (x, y) of the congruence

$$xy \equiv N \pmod{a}$$

are uniformly distributed as well.

Applications

It means that integer matrices such that

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = N$$

are uniformly distributed with respect to bi-invariant Haar measure on $GL_2(\mathbb{R})$.

Applications

It means that integer matrices such that

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = N$$

are uniformly distributed with respect to bi-invariant Haar measure on $GL_2(\mathbb{R})$.
This observation gives the way to study reduced bases in 2D lattices, continued fractions etc. Applications include:

Applications

It means that integer matrices such that

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = N$$

are uniformly distributed with respect to bi-invariant Haar measure on $GL_2(\mathbb{R})$.
This observation gives the way to study reduced bases in 2D lattices, continued fractions etc. Applications include:

- Gauss–Kuz'min statistics for rational numbers and quadratic irrationalities.
- Distribution of Frobenius numbers with 3 arguments.
- Distribution of free path lengths in 2D lattices (Lorenz gas).

The key idea

If

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = N,$$

then

The key idea

If
$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = N,$$
then we can fix a and consider a congruence
$$bc \equiv N \pmod{a}.$$

The key idea

If

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = N,$$

then we can fix *a* and consider a congruence

$$bc \equiv N \pmod{a}.$$

For each solution (b, c) a pair $(zb, z^{-1}c)$ where $zz^{-1} \equiv 1 \pmod{a}$ is also a solution.

The key idea

If

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = N,$$

then we can fix a and consider a congruence

$$bc \equiv N \pmod{a}.$$

For each solution (b, c) a pair $(zb, z^{-1}c)$ where $zz^{-1} \equiv 1 \pmod{a}$ is also a solution.
Propagating a solution we have 3 degrees of freedom (u, t, $z \in \mathbb{Z}$):

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = N \quad \Rightarrow \quad \begin{vmatrix} a & zb + ua \\ z^{-1}c + ta & * \end{vmatrix} = N$$

3-dimensional case ($3 \to 2$-reduction)

Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $q = \det A \neq 0$ and

$$\begin{vmatrix} \multicolumn{2}{c}{\multirow{2}{*}{\Huge A}} & x_1 \\ & & x_2 \\ x_3 & x_4 & x_5 \end{vmatrix} = N.$$

Propagating a solution we have 5 degrees of freedom (u, v, s, t, $z \in \mathbb{Z}$):

3-dimensional case ($3 \to 2$-reduction)

Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $q = \det A \neq 0$ and

$$\begin{vmatrix} \multicolumn{2}{c}{\multirow{2}{*}{A}} & x_1 \\ & & x_2 \\ x_3 & x_4 & x_5 \end{vmatrix} = N.$$

Propagating a solution we have 5 degrees of freedom (u, v, s, t, $z \in \mathbb{Z}$):

$$\begin{vmatrix} \multicolumn{2}{c}{\multirow{2}{*}{A}} & z^{-1}x_1 + ua + vb \\ & & z^{-1}x_2 + uc + vd \\ zx_3 + sa + tc & zx_4 + sb + td & * \end{vmatrix} = P,$$

where $zz^{-1} \equiv 1 \pmod{q}$.
Averaging over z we can apply Kloosterman sums again.

3-dimensional case ($3 \to 2$-reduction)

Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $q = \det A \neq 0$ and

$$\begin{vmatrix} \multicolumn{2}{c}{\multirow{2}{*}{\Huge A}} & x_1 \\ & & x_2 \\ x_3 & x_4 & x_5 \end{vmatrix} = N.$$

Propagating a solution we have 5 degrees of freedom (u, v, s, t, $z \in \mathbb{Z}$):

$$\begin{vmatrix} \multicolumn{2}{c}{\multirow{2}{*}{\Huge A}} & z^{-1}x_1 + ua + vb \\ & & z^{-1}x_2 + uc + vd \\ zx_3 + sa + tc & zx_4 + sb + td & * \end{vmatrix} = P,$$

where $zz^{-1} \equiv 1 \pmod{q}$.
Averaging over z we can apply Kloosterman sums again.
Linnik and Skubenko (1964) used this argument for studying distribution of points on a variety defined by equation $\det(x_{ij}) = N$ $(i, j = 1, 2, 3)$.

The goal

1. to create a 3D analytical tool based on Linnik-Skubenko reduction;
2. apply this tool at least in one problem.

The basic problem

Let $\ell(a/b)$ be a length of continued fraction expansion for a/b.

Theorem (Heilbronn, 1968)

$$\frac{1}{\varphi(N)} \sum_{\substack{1\leqslant a\leqslant N\\(a,N)=1}} \ell(a/N) = \frac{2\log 2}{\zeta(2)} \log N + O(\log^4 \log N).$$

Theorem (Porter, 1975)

$$\frac{1}{\varphi(N)} \sum_{\substack{1\leqslant a\leqslant N\\(a,N)=1}} \ell(a/N) = \frac{2\log 2}{\zeta(2)} \log N + C_P + O(N^{-1/6+\varepsilon}),$$

$$C_P = \frac{2\log 2}{\zeta(2)} \left(\frac{3\log 2}{2} + 2\gamma - 2\frac{\zeta'(2)}{\zeta(2)} - 1 \right) - \frac{1}{2}.$$

Gauss–Kuz'min statistics

For rational $r = [a_0; a_1, \ldots, a_s]$ and real $x, y \in [0,1]$ Gauss–Kuz'min statistics $\ell(x,y)$ can be defined as follows

$$\ell_{x,y}(r) = \left|\left\{1 \leqslant j \leqslant \ell + 1 : [0; a_j, \ldots, a_\ell] \leqslant x, [0; a_{j-1}, \ldots, a_1] \leqslant y\right\}\right|.$$

Theorem (The generalization of Porter's theorem)

$$\frac{1}{\varphi(N)} \sum_{\substack{1 \leqslant a \leqslant N \\ (a,N)=1}} \ell_{x,y}(a/N) = \frac{2\log(1+xy)}{\zeta(2)} \log N + C_P(x,y) + O(N^{-1/6+\varepsilon}).$$

The leading coefficient is a Gauss measure of corresponding box:

$$\log(1+xy) = \int_0^x \int_0^y \frac{d\alpha\, d\beta}{(1+\alpha\beta)^2} = \mu(Box = [0,x] \times [0,y])$$

The Gaussian measure

$$d\mu = \frac{dx\,dy}{\begin{vmatrix} 1 & x \\ -y & 1 \end{vmatrix}^2}$$

is a (right) Haar measure on quotient space $D_2(\mathbb{R})\backslash GL_2(\mathbb{R})$.

The Gaussian measure

$$d\mu = \frac{dx\,dy}{\begin{vmatrix} 1 & x \\ -y & 1 \end{vmatrix}^2}$$

is a (right) Haar measure on quotient space $D_2(\mathbb{R})\backslash GL_2(\mathbb{R})$.

The 3D analogue of Gaussian measure

$$d\mu = \frac{dx_2\,dx_3\,dy_1\,dy_3\,dz_1\,dz_2}{\begin{vmatrix} 1 & x_2 & \pm x_3 \\ -y_1 & 1 & y_3 \\ z_1 & -z_2 & 1 \end{vmatrix}^3}.$$

is a (right) Haar measure on quotient space $D_3(\mathbb{R})\backslash GL_3(\mathbb{R})$.

The Gaussian measure

$$d\mu = \frac{dx\,dy}{\begin{vmatrix} 1 & x \\ -y & 1 \end{vmatrix}^2}$$

is a (right) Haar measure on quotient space $D_2(\mathbb{R})\backslash GL_2(\mathbb{R})$.

The 3D analogue of Gaussian measure

$$d\mu = \frac{dx_2\,dx_3\,dy_1\,dy_3\,dz_1\,dz_2}{\begin{vmatrix} 1 & x_2 & \pm x_3 \\ -y_1 & 1 & y_3 \\ z_1 & -z_2 & 1 \end{vmatrix}^3}.$$

is a (right) Haar measure on quotient space $D_3(\mathbb{R})\backslash GL_3(\mathbb{R})$.

The same measure describes behavior of Klein polyhedrons. The difference is in measure space. Measure space varies for different types of 3D continued fractions.

The main result

The main result is a 3D analogue of Porter's theorem.

Theorem (AU, 2015?)

Average (over primitive lattices $\Lambda \subset \mathbb{Z}^3$ with $\det \Lambda = N$) number of elements in 3D continued fraction is

$$c_2 \log^2 N + c_1 \log N + c_0 + O(N^{-1/21+\varepsilon}).$$

The main result

The main result is a 3D analogue of Porter's theorem.

Theorem (AU, 2015?)

Average (over primitive lattices $\Lambda \subset \mathbb{Z}^3$ with $\det \Lambda = N$) number of elements in 3D continued fraction is

$$c_2 \log^2 N + c_1 \log N + c_0 + O(N^{-1/21+\varepsilon}).$$

This theorem has a natural generalization on 3D Gauss — Kuz'min statistics. In this case the leading coefficient $c_2 = \mu(Box)$, $Box \subset [0,1]^6$.

The main result

The main result is a 3D analogue of Porter's theorem.

Theorem (AU, 2015?)

Average (over primitive lattices Λ with $\det\Lambda = N$) number of elements in 3D continued fraction is

$$c_2 \log^2 N + c_1 \log N + c_0 + O(N^{-1/21+\varepsilon}).$$

The lattice with basis matrix

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$$

is primitive iff

$$(X_1, X_2, X_3) = (Y_1, Y_2, Y_3) = (Z_1, Z_2, Z_3) = 1.$$

The main result

The main result is a 3D analogue of Porter's theorem.

Theorem (AU, 2015?)

Average (over primitive lattices Λ with $\det\Lambda = N$) number of elements in 3D continued fraction is

$$c_2 \log^2 N + c_1 \log N + c_0 + O(N^{-1/21+\varepsilon}).$$

The lattice with basis matrix

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$$

is primitive iff

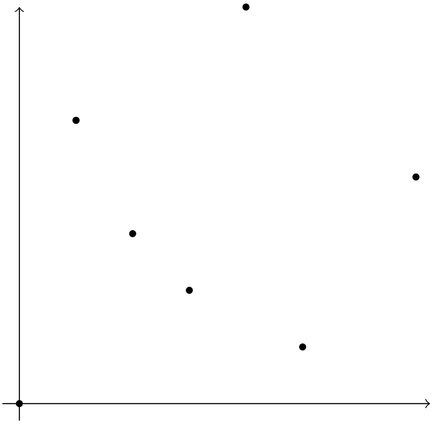
$$(X_1, X_2, X_3) = (Y_1, Y_2, Y_3) = (Z_1, Z_2, Z_3) = 1.$$

(In 2D case we considered a/N such that $(a, N) = 1$.)

Let S be a subset of $\mathbb{R}^2_{\geqslant 0}$. Consider the boundary of the set

$$S \oplus \mathbb{R}^2_{\geqslant 0} = \{s + r \mid s \in S, r \in \mathbb{R}^2_{\geqslant 0}\}.$$

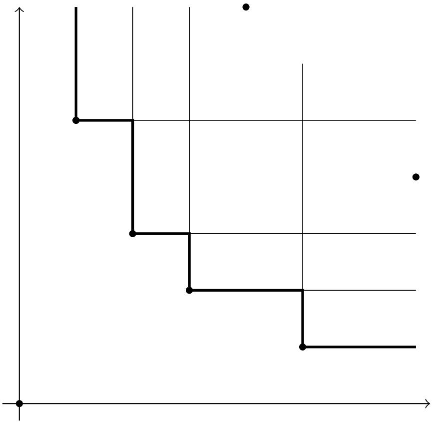
In other words, this broken line is the boundary of the union of copies of the positive quadrant shifted by vertices of the set S.



Let S be a subset of $\mathbb{R}^2_{\geqslant 0}$. Consider the boundary of the set

$$S \oplus \mathbb{R}^2_{\geqslant 0} = \{s + r \mid s \in S, r \in \mathbb{R}^2_{\geqslant 0}\}.$$

In other words, this broken line is the boundary of the union of copies of the positive quadrant shifted by vertices of the set S.



Let S be a subset of $\mathbb{R}^3_{\geqslant 0}$. The *Voronoi-Minkowski polyhedron* for S is the boundary of the set

$$S \oplus \mathbb{R}^3_{\geqslant 0} = \{s + r \mid s \in S, r \in \mathbb{R}^3_{\geqslant 0}\}.$$

In other words, the Voronoi-Minkowski polyhedron is the boundary of the union of copies of the positive octant shifted by vertices of the set S.

Let S be a subset of $\mathbb{R}^3_{\geqslant 0}$. The *Voronoi-Minkowski polyhedron* for S is the boundary of the set

$$S \oplus \mathbb{R}^3_{\geqslant 0} = \{s + r \mid s \in S, r \in \mathbb{R}^3_{\geqslant 0}\}.$$

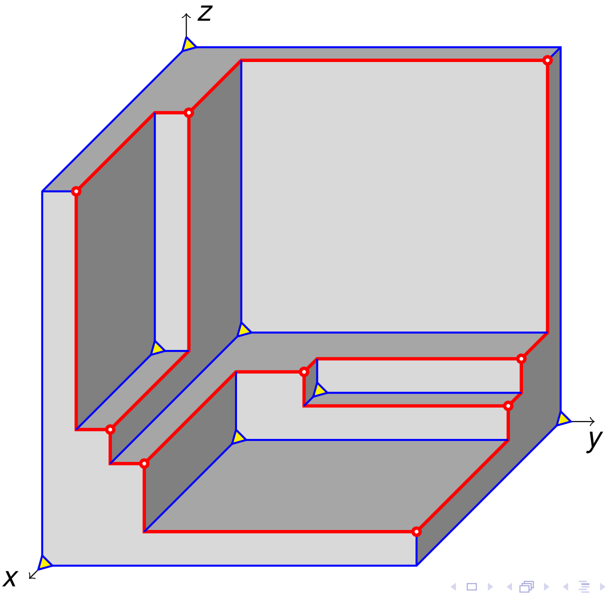
In other words, the Voronoi-Minkowski polyhedron is the boundary of the union of copies of the positive octant shifted by vertices of the set S.

We assume that

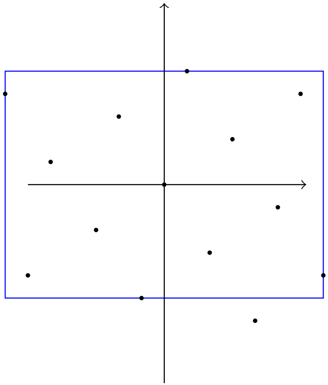
(1) S has no accumulation points;

(2) S is in general position: each plane parallel to a coordinate plane contains at most one point of S.

Voronoi-Minkowski complex



For a nonempty finite point set $T \subset \mathbb{R}^s$ $\mathrm{Box}(T)$ is the least possible parallelepiped circumscribed about T.



More formally: if

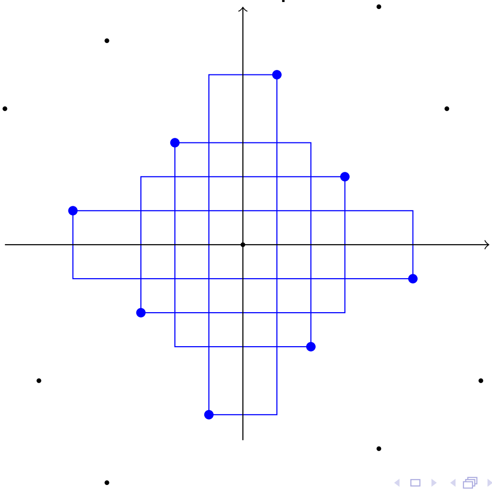
$$|T|_i = \max\{|x_i| : \ x = (x_1, \ldots, x_s) \in T\} \quad (i = 1, \ldots, s),$$

then

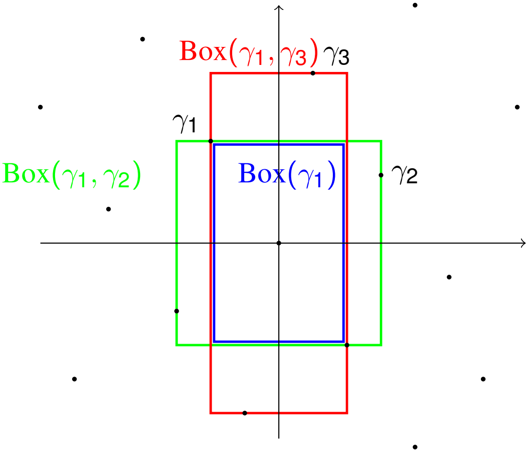
$$\mathrm{Box}(T) = [-|T|_1, |T|_1] \times \ldots \times [-|T|_s, |T|_s].$$

A point γ in a lattice Γ is called a *relative (local) minimum* of the lattice Γ in the sense of Voronoi (or simply a *minimum*) if the $\mathrm{Box}(\gamma)$ is *free* (it contains no points of the lattice Γ different from its vertices and the origin).

2D example:



The $\mathrm{Box}(\gamma_1, \gamma_2)$ is called *extreme* if it is *free* and if, at the same time, it has on each of its faces at least one lattice point.
In other words it is impossible to extend this parallelepiped in any coordinate direction so that the resulting parallelepiped still contains no nonzero lattice points.



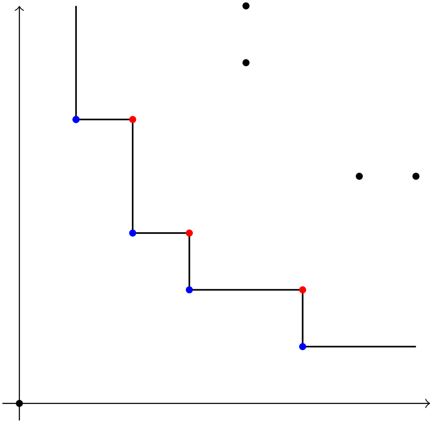
When we consider local minima or extreme parallelepipeds signs are not important for us. We can remove them.

When we consider local minima or extreme parallelepipeds signs are not important for us. We can remove them.

Instead of lattice Γ we can consider a set $|\Gamma| \subset \mathbb{R}^s$ where

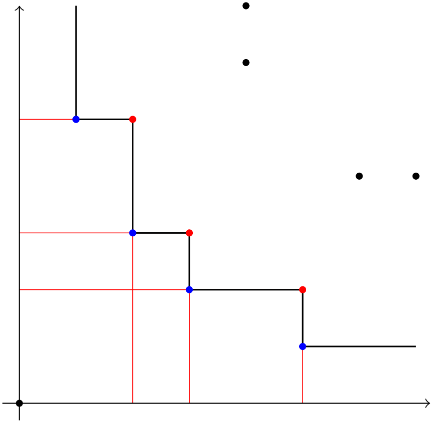
$$|\Gamma| = \{(|x|, |y|, |z|) : (x, y, z) \in \Gamma\}.$$

As it was proved by Voronoi, we can consider a classical continued fraction as a sequence of local minima (halls) or extreme parallelepipeds (hills)

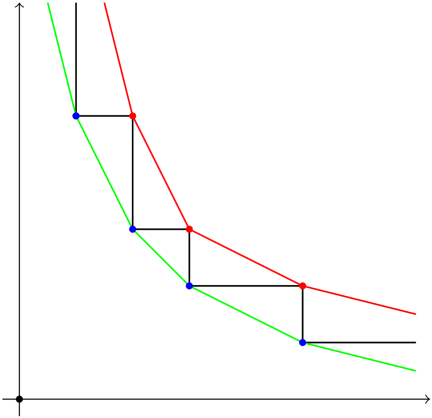


$\alpha \to \Gamma(\alpha) = \langle (1,0), (\alpha, 1) \rangle.$

As it was proved by Voronoi, we can consider a classical continued fraction as a sequence of local minima (halls) or extreme parallelepipeds (hills)

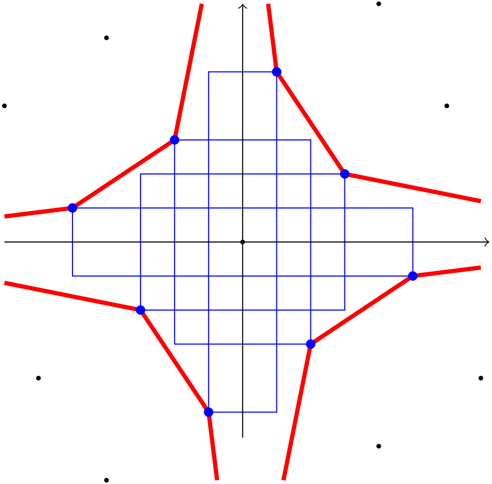


As it was proved by Voronoi, we can consider a classical continued fraction as a sequence of local minima (halls) or extreme parallelepipeds (hills)



Local minima and Klein polyhedron: (in 2D case)

local minima=vertices of Klein polyhedron



In 3D case vertices of Klein polyhedron are always local minima, but converse is not true (Bykovski, 2006).
In other words local minima have more rich structure (they can hide bihind the faces of Klein polyhedron.

3D definitions

The $\mathrm{Box}(\gamma_1, \gamma_2, \gamma_3)$ is called *extreme* if it is *free* (it contains no lattice points other than the origin) and if, at the same time, it has on each of its faces at least one lattice point.

3D definitions

The $\mathrm{Box}(\gamma_1, \gamma_2, \gamma_3)$ is called *extreme* if it is *free* (it contains no lattice points other than the origin) and if, at the same time, it has on each of its faces at least one lattice point.

It is impossible to extend this parallelepiped in any coordinate direction so that the resulting parallelepiped still contains no nonzero lattice points.

3D definitions

The $\mathrm{Box}(\gamma_1, \gamma_2, \gamma_3)$ is called *extreme* if it is *free* (it contains no lattice points other than the origin) and if, at the same time, it has on each of its faces at least one lattice point.

It is impossible to extend this parallelepiped in any coordinate direction so that the resulting parallelepiped still contains no nonzero lattice points.

A set of vectors (s.t. $v_i \neq v_j$) S in the lattice Γ is said to be *minimal* if the $\mathrm{Box}(S)$ contains no points of Γ except the origin. In particular, a minimal system of order 1 is a local minimum, minimal systems of order 3 gives extreme parallelepiped.

3D definitions

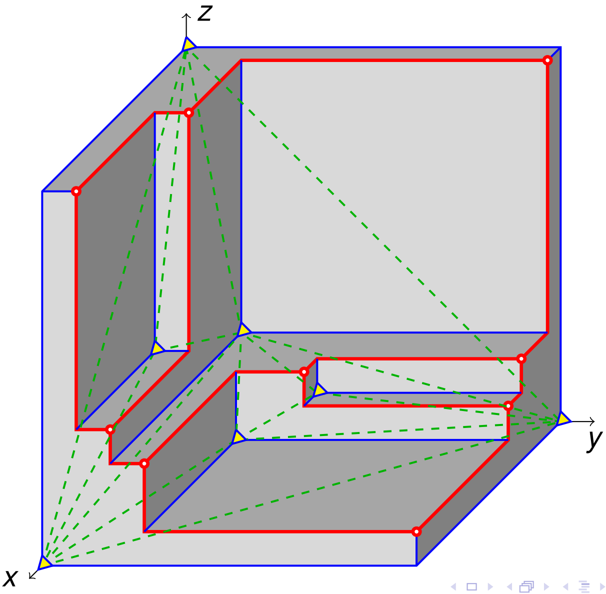
The $\mathrm{Box}(\gamma_1, \gamma_2, \gamma_3)$ is called *extreme* if it is *free* (it contains no lattice points other than the origin) and if, at the same time, it has on each of its faces at least one lattice point.

It is impossible to extend this parallelepiped in any coordinate direction so that the resulting parallelepiped still contains no nonzero lattice points.

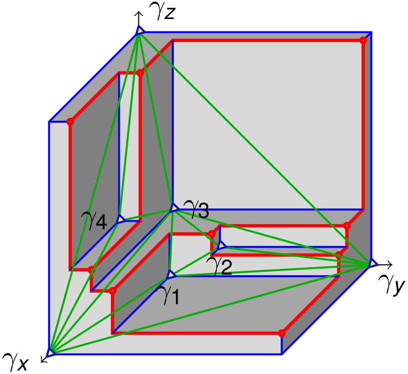
A set of vectors (s.t. $v_i \neq v_j$) S in the lattice Γ is said to be *minimal* if the $\mathrm{Box}(S)$ contains no points of Γ except the origin. In particular, a minimal system of order 1 is a local minimum, minimal systems of order 3 gives extreme parallelepiped.

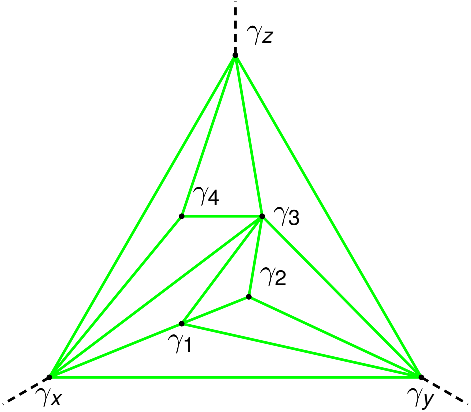
If $\{\gamma_1, \gamma_2\}$ is a minimal system of order 2 then γ_1 and γ_2 are *neighbours*.

Minkowski graph

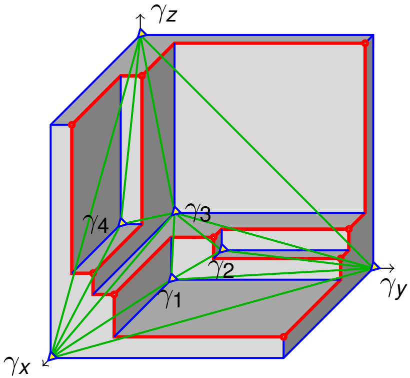


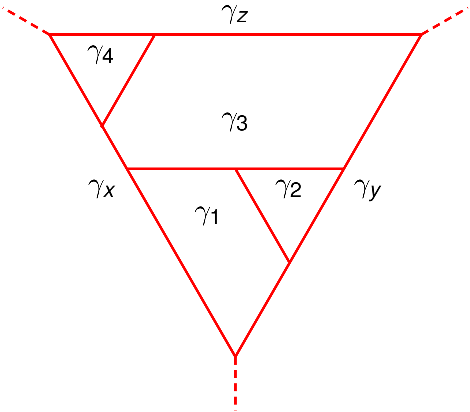
Minkowski graph



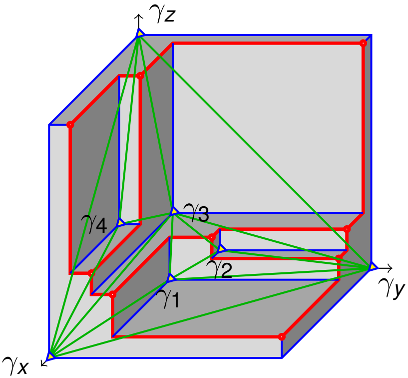


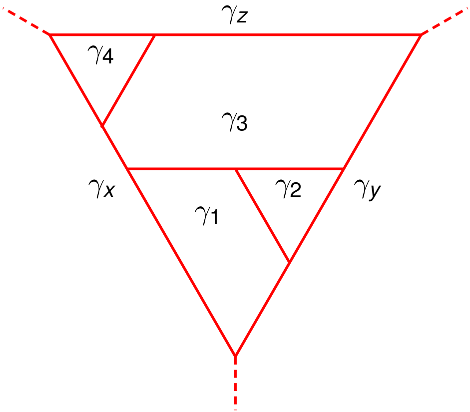
Voronoi (=Minkowski*) graph





Voronoi (=Minkowski*) graph





For more pictures and explanations see

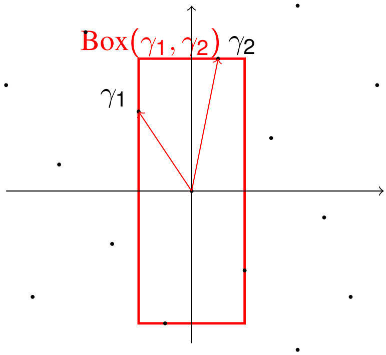
O. KARPENKOV, A. USTINOV “Geometry of Minkowski-Voronoi tessellations of the plane” (ArXiv e-prints. Submitted on 1 Jul 2014).

Some reasons

Why do this objects are interesting and important?

- Good algorithms.
- Periodicity for algebraic lattices.
- “Vahlen’s theorem”.
- “Gauss measure”.
- Possibility to apply “hard” (analytical) methods based on Kloosterman sums.

In 2D case minimal couple $\gamma_1 = (a_1, b_1)$, $\gamma_2 = (a_2, b_2)$ is always a basis of a given lattice (Voronoi):



Some reasons

Gauss measure

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \to \begin{pmatrix} a_1^{-1} & 0 \\ 0 & b_2^{-1} \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = \begin{pmatrix} 1 & x \\ -y & 1 \end{pmatrix}$$

where $0 < x < 1$, $0 < y < 1$. $\mathrm{Box}(\gamma_1, \gamma_2) \to [-1, 1]^2$.

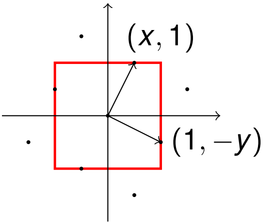
Some reasons

Gauss measure

Gaussian measure

$$d\mu = \frac{dx\,dy}{(1+xy)^2} = \frac{dxdy}{\begin{vmatrix} 1 & x \\ -y & 1 \end{vmatrix}^2}$$

defined for $(x, y) \in [0, 1]^2$ describes typical behavior of classical continued fractions. From geometrical point of view this density function describes distribution of vectors from bases $\left(\begin{smallmatrix} 1 & x \\ -y & 1 \end{smallmatrix}\right)$ on the sides of unit square.



Some reasons

Gauss measure

In 2D case minimal couple $\gamma_1 = (a_1, b_1)$, $\gamma_2 = (a_2, b_2)$ is always a basis of a given lattice and $\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \sim \begin{pmatrix} 1 & x \\ -y & 1 \end{pmatrix}$ where $0 < x < 1$, $0 < y < 1$.

Some reasons

Gauss measure

In 2D case minimal couple $\gamma_1 = (a_1, b_1)$, $\gamma_2 = (a_2, b_2)$ is always a basis of a given lattice and $\left(\begin{smallmatrix} a_1 & a_2 \\ b_1 & b_2 \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} 1 & x \\ -y & 1 \end{smallmatrix}\right)$ where $0 < x < 1$, $0 < y < 1$.

3D surprise (Minkowski): either minimal triple $\gamma_1 = (a_1, b_1, c_1)$, $\gamma_2 = (a_2, b_2, c_2)$, $\gamma_3 = (a_3, b_3, c_3)$ is a basis and corresponding matrix equivalent to

$$\begin{pmatrix} 1 & x_2 & \pm x_3 \\ -y_2 & 1 & y_3 \\ z_1 & -z_2 & 1 \end{pmatrix}$$

In 2D case minimal couple $\gamma_1 = (a_1, b_1)$, $\gamma_2 = (a_2, b_2)$ is always a basis of a given lattice and $\left(\begin{smallmatrix} a_1 & a_2 \\ b_1 & b_2 \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} 1 & x \\ -y & 1 \end{smallmatrix}\right)$ where $0 < x < 1$, $0 < y < 1$.

3D surprise (Minkowski): either minimal triple $\gamma_1 = (a_1, b_1, c_1)$, $\gamma_2 = (a_2, b_2, c_2)$, $\gamma_3 = (a_3, b_3, c_3)$ is a basis and corresponding matrix equivalent to

$$\begin{pmatrix} 1 & x_2 & \pm x_3 \\ -y_2 & 1 & y_3 \\ z_1 & -z_2 & 1 \end{pmatrix}$$

or it is degenerate ($\det(\gamma_1, \gamma_2, \gamma_3) = 0$) and for some combination of signs

$$\gamma_1 \pm \gamma_2 \pm \gamma_3 = 0.$$

The 3D analogue of Gaussian measure

$$d\mu = \frac{dx_2\, dx_3\, dy_1\, dy_3\, dz_1\, dz_2}{\begin{vmatrix} 1 & x_2 & \pm x_3 \\ -y_2 & 1 & y_3 \\ z_1 & -z_2 & 1 \end{vmatrix}^3}$$

describes a distribution of basis vectors on some subset of $[0, 1]^6$ (defined by some simple liner inequalities).

Two main examples (the beginning of *Markov spectrum*) are

$$\frac{1+\sqrt{5}}{2} = 2\cos\frac{2\pi}{5} = [1;1,\ldots,1,\ldots] = 1+\cfrac{1}{1+\ddots+\cfrac{1}{1+\ldots}},$$

$$\sqrt{2} = 2\cos\frac{2\pi}{8} = [1;2,\ldots,2,\ldots] = 1+\cfrac{1}{2+\ddots+\cfrac{1}{2+\ldots}}.$$

Some reasons

Periodicity

With quadratic irrational α we can associate a lattice $\Gamma(\alpha)$ with basis $(1, 1)$ and (α, β) where β is conjugate of α (second root of the same quadratic equation.)
Periodical continued fraction of α describes periodical structure of local minima of $\Gamma(\alpha)$.

With quadratic irrational α we can associate a lattice $\Gamma(\alpha)$ with basis $(1, 1)$ and (α, β) where β is conjugate of α (second root of the same quadratic equation.)

Periodical continued fraction of α describes periodical structure of local minima of $\Gamma(\alpha)$.

With cubic irrationality α (from totally real cubic field) we can associate 3D lattice with basis $(1, 1, 1)$, (α, β, γ), $(\alpha^2, \beta^2, \gamma^2)$, where β and γ are conjugates of α.

Some reasons

Periodicity

With quadratic irrational α we can associate a lattice $\Gamma(\alpha)$ with basis $(1, 1)$ and (α, β) where β is conjugate of α (second root of the same quadratic equation.)

Periodical continued fraction of α describes periodical structure of local minima of $\Gamma(\alpha)$.

With cubic irrationality α (from totally real cubic field) we can associate 3D lattice with basis $(1, 1, 1)$, (α, β, γ), $(\alpha^2, \beta^2, \gamma^2)$, where β and γ are conjugates of α.

Voronoi–Minkowski graph for such lattices is doubly periodic (totally real cubic field has 2 fundamental units).

Some reasons

Periodicity

With quadratic irrational α we can associate a lattice $\Gamma(\alpha)$ with basis $(1,1)$ and (α,β) where β is conjugate of α (second root of the same quadratic equation.)

Periodical continued fraction of α describes periodical structure of local minima of $\Gamma(\alpha)$.

With cubic irrationality α (from totally real cubic field) we can associate 3D lattice with basis $(1,1,1)$, (α,β,γ), $(\alpha^2,\beta^2,\gamma^2)$, where β and γ are conjugates of α.

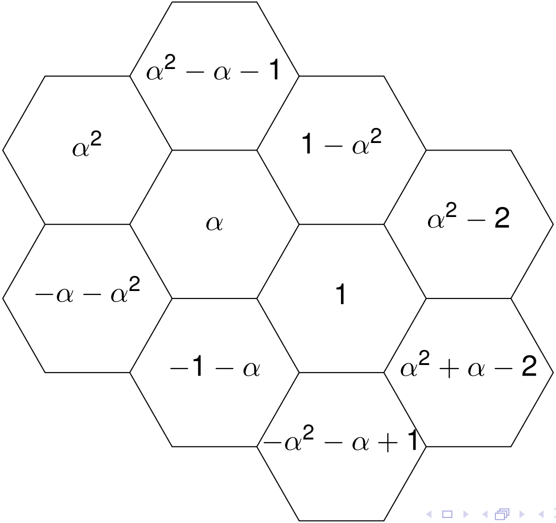
Voronoi–Minkowski graph for such lattices is doubly periodic (totally real cubic field has 2 fundamental units).

Two mains examples arise from cubic numbers $\alpha = 2\cos\frac{2\pi}{7}$ and $\alpha = 2\cos\frac{2\pi}{9}$ (associated with first two *extremal Davenport cubic forms*).

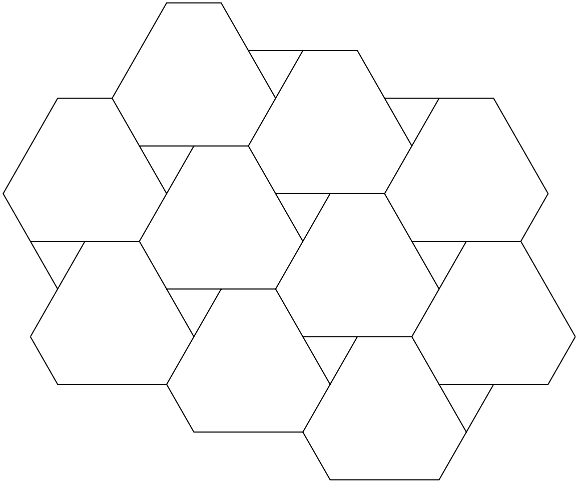
Some reasons

Periodicity

The Voronoi graph for $\alpha = 2\cos\frac{2\pi}{7}$



The Voronoi graph for $\alpha = 2\cos\frac{2\pi}{9}$



SYTA H., VAN DE WEYGAERT R. "Life and Times of Georgy Voronoï" (ArXiv e-prints, 2009).

"Markov asked Voronoï by telegraph to come from Warsaw to Petrograd. Markov invited Voronoï to his office and proposed him to calculate the unit for the cubic equation $r^3 = 23$. By artificial means, Markov had found for this example the unit

$$e = 2166673601 + 761875860r + 267901370r^2.$$

Voronoï calculated for three hours.

The period had 21 terms and in order to find the main unit it was necessary to multiply 21 expressions

$$-2+\rho,\ \frac{-11+2\rho+\rho^2}{15},\ \frac{-3-\rho+\rho^2}{4},\ \frac{-9+5\rho+\rho^2}{17},\ \frac{4-3\rho+\rho^2}{10},$$
$$\frac{1-\rho+\rho^2}{8},\ \frac{-2+\rho}{3},\ \frac{1+3\rho-\rho^2}{10},\ \frac{-5-\rho+\rho^2}{3},\ \frac{-1+\rho}{2},$$
$$\frac{-10+\rho+\rho^2}{11},\ -2+\rho,\ \frac{-11+2\rho+\rho^2}{15},\ \frac{1-\rho+\rho^2}{8},\ \frac{-2+\rho}{3},$$
$$\frac{1+3\rho-\rho^2}{5},\ \frac{-1+\rho}{2},\ \frac{-1+10\rho-\rho^2}{33},\ \frac{-11+7\rho+\rho^2}{20},$$
$$\frac{9-7\rho+2\rho}{31},\ \frac{-5-\rho+\rho^2}{6}.$$

Following this analysis, he found the unit

$$E = -41399 - 3160r + 6230r^2.$$

It turned out that $Ee = 1$. So, it was verified that the algorithm really worked."
Markov's unit one more time:

$$e = 2166673601 + 761875860r + 267901370r^2.$$

Thank you for your attention!