Distribution of Frobenius Numbers

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Let $a_1, \ldots, a_n$ be positive integers with $a_i \geq 2$ and $(a_1, \ldots, a_n) = 1$. The following naive questions is known as “Diophantine Frobenius problem” (or “Coin exchange problem”): Determine the largest number which is not of the form

$$a_1 x_1 + \cdots + a_n x_n$$

where the coefficients $x_i$ are non-negative integers. This number is denoted by $g(a_1, \ldots, a_n)$ and is called the **Frobenius number**.
Frobenius numbers
The Diophantine Frobenius problem

Example
Let $a = 3$, $b = 5$. Then $g(a, b) =$?
Frobenius numbers
The Diophantine Frobenius problem

Example
Let \(a = 3\), \(b = 5\). Then \(g(a, b) = 7\):

\[ 7 \neq 3x + 5y \quad (x, y \geq 0), \]

but for every \(m > 7\) there are some \(x, y \geq 0\) such that \(m = 3x + 5y\).
Example

Let $a = 3$, $b = 5$. Then $g(a, b) = 7$:

$$7 \neq 3x + 5y \quad (x, y \geq 0),$$

but for every $m > 7$ there are some $x, y \geq 0$ such that $m = 3x + 5y$.

It is known that

$$g(a, b) = ab - a - a.$$

The challenge is to find $g(a_1, \ldots, a_n)$ when $n \geq 3$. 
Frobenius numbers
The Diophantine Frobenius problem

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The challenge is to find $g(a_1, \ldots, a_n)$ when $n \geq 3$.

Example
$g(3, 5, 7) = 4$:

$$4 \neq 3x + 5y + 7z \quad (x, y, z \geq 0).$$
We shall consider

\[ f(a, b, c) = g(a, b, c) + a + b + c, \]

the **positive Frobenius number** of \(a, b, c\), defined to be the largest integer not representable as a **positive** linear combination of \(a, b, c\)

\[ ax + by + cz, \quad x, y, z \geq 1. \]

Positive Frobenius numbers are better because of Johnson’s formula: for \(d \mid a, d \mid b\)

\[ f(a, b, c) = d \cdot f \left( \frac{a}{d}, \frac{b}{d}, c \right). \]
Double loop network

\[ b = 3 \text{ (red step)}, \quad c = 5 \text{ (blue step)}, \quad a = 7 \text{ (number of vertices)} \]

\[
\text{length}(↑) = 3, \quad \text{length}(↑′) = 5
\]

\[ t(x, y) = bx + cy \text{ (time)} \]

\[
\begin{array}{cccc}
5 & 8 & 11 & \\
0 & 3 & 6 & 9
\end{array}
\]

\[ n \equiv t \pmod{a} \text{ (number)} \]

\[
\begin{array}{cccc}
5 & 1 & 4 & \\
0 & 3 & 6 & 2
\end{array}
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### Network Diagram

#### 1. Diameter Calculation
\[
diam = g(a, b, c) + a \quad (= 11)
\]

#### 2. Time Calculation
\[
t(x, y) = bx + cy \quad \text{(time)}
\]

#### 3. Number Modulo Calculation
\[
n \equiv t \pmod{a} \quad \text{(number)}
\]
Double loop network

\( b = 9 \) (red step), \( c = 5 \) (blue step), \( a = 17 \) (number of vertices)

\[ \Lambda = \{ (x, y) : bx + cy \equiv 0 \pmod{a} \} \]

\((-c, b)\)
Double loop network

\( b = 9 \) (red step), \( c = 5 \) (blue step), \( a = 17 \) (number of vertices)

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\(-s_{m+1}, q_{m+1}\)

\(-c, b\)

\(-s_{v+1}, q_{v+1}\)

\(-s_v, q_v\)

\(-s_0, q_0\)
Double loop network

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\[ \Lambda = \{(x, y) : bx + cy \equiv 0 \pmod{a}\} \]
Rødseth’s formula

From obvious property

\[ 0 = \frac{s_{m+1}}{q_{m+1}} < \frac{s_{m-1}}{q_{m-1}} < \ldots < \frac{s_1}{q_1} < \frac{s_0}{q_0} = \infty \]

follows that for some \( n \)

\[ \frac{s_n}{q_n} \leq \frac{c}{b} < \frac{s_{n-1}}{q_{n-1}}. \]
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**Theorem (Ö. Rødseth, 1978)**

\[ f(a, b, c) = bs_{n-1} + cq_n - \min \{ bs_n, cq_{n-1} \}. \]
Rödseth’s formula can be written in terms of reduced regular continued fraction. We want to find $f(a, b, c)$ for $(a, b) = (a, c) = (b, c) = 1$. Let $l$ is such that

$$bl \equiv c \pmod{a}, \quad 1 \leq l \leq a.$$  

Reduced regular continued fraction

$$\frac{a}{l} = \langle a_1, \ldots, a_m \rangle = a_1 - \frac{1}{a_2 - \frac{1}{\ddots - \frac{1}{a_m}}}.$$  

where $a_1, \ldots, a_m \geq 2$, defines sequences $\{s_j\}, \{q_j\}$ by

$$\frac{q_{j+1}}{q_j} = \langle a_j, \ldots, a_1 \rangle, \quad \frac{s_j}{s_{j+1}} = \langle a_{j+1}, \ldots, a_m \rangle \quad (0 \leq j \leq m).$$
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\]
We have one-to-one correspondence between the set of quadruples $(q_n, s_n, q_{n-1}, s_{n-1})$ (taken for all lattices $\Lambda_l$) and the solutions of the equation

$$x_1 y_1 - x_2 y_2 = a$$

with $0 \leq x_2 < x_1$, $0 \leq y_2 < y_1$, $(x_1, x_2) = (y_1, y_2) = 1$:

$$(q_n, s_n, q_{n-1}, s_{n-1}) \leftrightarrow (x_1, x_2, y_2, y_1).$$
From the equation

\[ x_1 y_1 - x_2 y_2 = a \]

it follows that

\[ x_1 y_1 \equiv a \pmod{x_2}, \]

and Kloosterman sums

\[ K_q(l, m, n) = \sum_{\substack{x, y=1 \\ xy \equiv l \pmod{q}}} e^{2\pi i \frac{mx+ny}{q}} \]

come into play. Solutions of the congruence \( xy \equiv l \pmod{q} \) are uniformly distributed due to the bounds for Kloosterman sums.
This fact allows to calculate sums of the form

\[ \sum_{xy \equiv l \pmod{q}} F(x, y) \]

and

\[ \sum_{x_1y_1 - x_2y_2 = a} F(x_1, y_1, x_2, y_2). \]

In particular it allows to study distribution of Frobenius numbers \( f(a, b, c) \).
Rødseth (1990) proved a lower bound for Frobenius numbers:

\[ f(a_1, \ldots, a_n) \geq n^{-1}(n - 1)!a_1 \cdots a_n. \]

**Conjecture (Davison, 1994)**

Average value of normalized Frobenius numbers \( \frac{f(a,b,c)}{\sqrt{abc}} \) over cube \([1, N]^3\) tends to some constant as \( N \to \infty \).
Conjectures

Rødseth (1990) proved a lower bound for Frobenius numbers:

\[ f(a_1, \ldots, a_n) \geq n^{-1} \sqrt{(n-1)! a_1 \cdots a_n}. \]

Conjecture (Davison, 1994)

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Conjecture (Arnold, 1999, 2005)

There is weak asymptotic for Frobenius numbers: for arbitrary \( n \) average value of \( f(x_1, \ldots, x_n) \) over small cube with a center in \((a_1, \ldots, a_n)\) approximately equal to \( c_n n^{-1} \sqrt{a_1 \cdots a_n} \) for some constant \( c_n > 0 \).
Theorem (Bourgain and Sinaĭ, 2007)

Normalized Frobenius numbers \( \frac{f(a,b,c)}{\sqrt{abc}} \) (under some natural assumption) have limiting density function.
Weak asymptotic

Let $x_1, x_2 > 0$ and

$M_a(x_1, x_2) = \{(b, c) : 1 \leq b \leq x_1 a, 1 \leq c \leq x_2 a, (a, b, c) = 1\}.$
Weak asymptotic

Let $x_1, x_2 > 0$ and
\[ M_a(x_1, x_2) = \{(b, c) : 1 \leq b \leq x_1 a, 1 \leq c \leq x_2 a, (a, b, c) = 1\} \]

**Theorem (A.U., 2009)**

**Frobenius numbers** $f(a, b, c)$ have weak asymptotic $\frac{8}{\pi} \sqrt{abc}$:

\[
\frac{1}{a^{3/2}|M_a(x_1, x_2)|} \sum_{(b, c) \in M_a(x_1, x_2)} \left( f(a, b, c) - \frac{8}{\pi} \sqrt{abc} \right) = O_{\varepsilon, x_1, x_2}(a^{-1/6+\varepsilon}).
\]

**Davison’s conjecture holds in a stronger form:**

\[
\frac{1}{|M_a(x_1, x_2)|} \sum_{(b, c) \in M_a(x_1, x_2)} \frac{f(a, b, c)}{\sqrt{abc}} = \frac{8}{\pi} + O_{\varepsilon, x_1, x_2}(a^{-1/6+\varepsilon}).
\]
Theorem (A.U., 2010)

Normalized Frobenius numbers of three arguments have limiting density function:

\[
\frac{1}{|M_a(x_1, x_2)|} \sum_{(b,c) \in M_a(x_1, x_2) \atop f(a,b,c) \leq \tau \sqrt{abc}} 1 = \int_0^\tau p(t) \, dt + O_{\varepsilon, x_1, x_2, \tau}(a^{-1/6+\varepsilon}),
\]

where

\[
p(t) = \begin{cases} 
0, & \text{if } t \in [0, \sqrt{3}]; \\
\frac{12}{\pi} \left( \frac{t}{\sqrt{3}} - \sqrt{4 - t^2} \right), & \text{if } t \in [\sqrt{3}, 2]; \\
\frac{12}{\pi^2} \left( t \sqrt{3 \arccos \frac{t+3\sqrt{t^2-4}}{4\sqrt{t^2-3}}} + \frac{3}{2} \sqrt{t^2-4} \log \frac{t^2-4}{t^2-3} \right), & \text{if } t \in [2, +\infty). 
\end{cases}
\]
Density function

\[
\left(2, 8\sqrt{3}/\pi\right)
\]

\[
\lim_{t \to 2^{-}} p'(t) = +\infty, \quad \lim_{t \to 2^{+}} p'(t) = -\infty
\]

\[
p(t) = \frac{18}{\pi^3} \cdot \frac{1}{t^3} + O\left(\frac{1}{t^5}\right) \quad (t \to \infty)
\]

\[
\int_{0}^{\infty} p(t) \, dt = 1, \quad \int_{0}^{\infty} tp(t) \, dt = \frac{8}{\pi}
\]
Density function

Triples \( (\alpha, \beta, r) \), where

\[
\alpha = \frac{q_n}{\sqrt{a/x}}, \quad \beta = \frac{s_{n-1}}{\sqrt{a/\xi}}, \quad r = \frac{s_n}{\sqrt{a\xi}} \quad (\xi = c/b)
\]

(normalized edges of L-shaped diagram) have joint limiting density function

\[
p(\alpha, \beta, r) = \begin{cases} 
2 & r \leq \min\{\alpha, \beta\}, 1 \leq \alpha\beta \leq 1 + r^2, \\
\frac{\zeta(2)}{r} & \text{else.}
\end{cases}
\]

It allows to study shortest cycles, average distances and another characteristics of L-shaped diagrams (double loop networks).
Weak asymptotic for genus

Let

\[ n(a, b, c) = \#(\mathbb{N} \setminus \langle a, b, c \rangle) \]

be a genus of numerical semigroup \( \langle a, b, c \rangle \) and let \( N(a, b, c) \) let be modified genus:

\[ N(a, b, c) = n(a, b, c) + \frac{a}{2} + \frac{b}{2} + \frac{c}{2} - \frac{1}{2}. \]

It is more convenient because for \( d \mid a, d \mid b \) we have

\[ N(a, b, c) = d \cdot N\left(\frac{a}{d}, \frac{b}{d}, c\right). \]

Theorem (Vorob’ev, 2016)

\[ N(a, b, c) \approx \frac{64}{5\pi^2} \sqrt{abc}. \]
General idea
Kloosterman sums

For usual Kloosterman sums

\[ K_q(1, m, n) = \sum_{x, y=1}^{q} e^{2\pi i \frac{mx+ny}{q}} \]

xy \equiv 1 \pmod{q}

Estermann bound is known

\[ |K_q(1, m, n)| \leq \sigma_0(q) \cdot (m, n, q)^{1/2} \cdot q^{1/2}. \]

This bound can be generalized for the case of sums \( K_q(l, m, n) \).
General idea

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This bound can be generalized for the case of sums \( K_q(l, m, n) \).

Theorem (A.U., 2008)

\[ |K_q(l, m, n)| \leq \sigma_0(q) \cdot \sigma_0((l, m, n, q)) \cdot (lm, ln, mn, q)^{1/2} \cdot q^{1/2}. \]

This estimate allows to count solutions of the congruence \( xy \equiv l \pmod{a} \) in different regions.
Corollary

Let \( q \geq 1, \ 0 \leq P_1, P_2 \leq q \). Then for any real \( Q_1, Q_2 \)

\[
\sum_{Q_1 < x \leq Q_1 + P_1 \atop Q_2 < y \leq Q_2 + P_2} \delta_q(xy - 1) = \frac{\varphi(q)}{q^2} \cdot P_1 P_2 + O \left( \sigma_0(q) \log^2(q + 1)q^{1/2} \right)
\]

and

\[
\sum_{Q_1 < x \leq Q_1 + P_1 \atop Q_2 < y \leq Q_2 + P_2} \delta_q(xy - l) = \frac{K_q(0, 0, l)}{q^2} \cdot P_1 P_2 + O \left( q^{1/2+\epsilon} + (q, l)q^\epsilon \right).
\]
A combination with *van der Corput’s method* of exponential sums allows to count solutions under a graph of smooth function.
A combination with van der Corput’s method of exponential sums allows to count solutions under a graph of smooth function. Let \( q \geq 1 \), \( f \) be positive function and \( T[f] \) be the number of solutions of the congruence \( xy \equiv l \pmod{q} \) in the region \( P_1 < x \leq P_2 \), \( 0 < y \leq f(x) \): 

\[
T[f] = \sum_{P_1 < x \leq P_2} \sum_{0 < y \leq f(x)} \delta_q(xy - l).
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A combination with **van der Corput’s method** of exponential sums allows to count solutions under a graph of smooth function.

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$$T[f] = \sum_{P_1 < x \leq P_2} \sum_{0 < y \leq f(x)} \delta_q(xy - l).$$

Let

$$S[f] = \sum_{P_1 < x \leq P_2} \frac{\mu_{q,l}(x)}{q} f(x),$$

where $\mu_{q,l}(x)$ is the number of solutions of the congruence $xy \equiv l \pmod{q}$ over $y$ such that $1 \leq y \leq q$. 
Theorem (A.U., 2008)

Let $P_1, P_2$ be reals, $P = P_2 - P_1 \geq 2$ and for some $A > 0$, $w \geq 1$ function $f(x)$ satisfies conditions

$$\frac{1}{A} \leq |f''(x)| \leq \frac{w}{A}.$$

Then

$$T[f] = S[f] - \frac{P}{2} \cdot \delta_q(l) + R[f],$$

where

$$R[f] \ll_w (PA^{-1/3} + A^{1/2}(l, q)^{1/2} + q^{1/2})P^\varepsilon.$$
Recent results

- The existence of limiting distribution for normalized Frobenius numbers of arbitrary number of arguments was proved by J. Marklof (2010).

- Distribution of diameters and distribution of shortest cycles in circulant graphs (often also called multi-loop networks) were studied by J. Marklof and A. Strömbergsson (2011). They proved existence of these distributions for arbitrary $n$ and made some interesting numerical computations.

- For $n = 3$ Davison’s conjecture in a stronger form was proved by D. Frolenkov (2011).

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Let $1 \leq l \leq a$, $(l, a) = 1$ and $e_1$ be the shortest vector of the lattice
\[ \Lambda_l = \{(x, y) : lx \equiv y \pmod{a}\}. \]
Let $1 \leq l \leq a$, $(l, a) = 1$ and $e_1$ be the shortest vector of the lattice \( \Lambda_l = \{(x, y) : lx \equiv y \pmod{a}\} \). Basis \((e_1, e_2)\) is reduced iff $e_2 \in \Omega(e_1)$ where \( \Omega(e_1) \) is the plane region defined by inequalities

\[
\|e_2\| \geq \|e_1\| \quad \text{and} \quad \|e_2 \pm e_1\| \geq \|e_2\|.
\]
Let $1 \leq l \leq a$, $(l, a) = 1$ and $e_1$ be the shortest vector of the lattice $\Lambda_l = \{(x, y) : lx \equiv y \pmod{a}\}$. Basis $(e_1, e_2)$ is reduced iff $e_2 \in \Omega(e_1)$ where $\Omega(e_1)$ is the plane region defined by inequalities

$$\|e_2\| \geq \|e_1\| \quad \text{and} \quad \|e_2 \pm e_1\| \geq \|e_2\|.$$

Moreover vector $e_2$ must lie on the line $l(e_1)$ defined by equation $\det(e_1, e_2) = a$. 
Let $1 \leq l \leq a$, $(l, a) = 1$ and $e_1$ be the shortest vector of the lattice $\Lambda_l = \{(x, y) : lx \equiv y \pmod{a}\}$. Basis $(e_1, e_2)$ is reduced iff $e_2 \in \Omega(e_1)$ where $\Omega(e_1)$ is the plane region defined by inequalities

$$\|e_2\| \geq \|e_1\| \quad \text{and} \quad \|e_2 \pm e_1\| \geq \|e_2\|.$$  

Moreover vector $e_2$ must lie on the line $l(e_1)$ defined by equation $\det(e_1, e_2) = a$. By averaging over $l$ we can get that vectors $e_2$ distributed uniformly on $\Omega(e_1) \cap l(e_1)$ with weight $\|e_2\|^{-1}$. Suppose $e_1 = \sqrt{a}(\alpha, \beta)$, $e_2 = \sqrt{a}(\gamma, \delta)$. 
Reduced bases in two-dimensional lattices

By integrating over $e_1$ we can get density function for $t = \|e_2\|/\sqrt{a}$:

$$p(t) = \begin{cases} 
0, & \text{if } t \in [0, 1/\sqrt{2}] \\
4\zeta(2)(2t - 1)t + (1 - t^2)\log(1 - t^2), & \text{if } t \in [1/\sqrt{2}, 1] \\
4\zeta(2)(1 + (t - 1)t)\log((1 - t^2), & \text{if } t \in [1, \infty) 
\end{cases}$$
By integrating over $e_1$ we can get density function for $t = \|e_2\|/\sqrt{a}$:

$$p(t) = \begin{cases} 
0, & \text{if } t \in \left[0, \frac{1}{\sqrt{2}}\right]; \\
\frac{4}{\zeta(2)} \left(2t - \frac{1}{t} + \left(\frac{1}{t} - t\right) \log \left(\frac{1}{t^2} - 1\right)\right), & \text{if } t \in \left[\frac{1}{\sqrt{2}}, 1\right]; \\
\frac{4}{\zeta(2)} \left(\frac{1}{t} + (t - \frac{1}{t}) \log \left(1 - \frac{1}{t^2}\right)\right), & \text{if } t \in [1, \infty].
\end{cases}$$
Reduced bases in two-dimensional lattices

By integrating over $e_1$, we can get density function for $t = \|e_2\|/\sqrt{a}$:

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\frac{4}{\zeta(2)} \left( 2t - \frac{1}{t} + \left( \frac{1}{t} - t \right) \log \left( \frac{1}{t^2} - 1 \right) \right), & \text{if } t \in \left[1/\sqrt{2}, 1\right]; \\
\frac{4}{\zeta(2)} \left( \frac{1}{t} + (t - \frac{1}{t}) \log \left( 1 - \frac{1}{t^2} \right) \right), & \text{if } t \in [1, \infty]. 
\end{cases}$$


Questione?