Izvestiya: *Mathematics* **73**:4 669–688 © 2009 RAS(DoM) and LMS
Izvestiya RAN: *Ser. Mat.* **73**:4 17–36 DOI 10.1070/IM2009v073n04ABEH002461

The statistics of particle trajectories in the inhomogeneous Sinai problem for a two-dimensional lattice

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Abstract. In connection with the two-dimensional model known as the 'periodic Lorentz gas', we study the asymptotic behaviour of statistical characteristics of a free path interval of a point particle before its first occurrence in an h-neighbourhood (a circle of radius h) of a non-zero integer point as $h \to 0$ given that the particle starts from the h-neighbourhood of the origin. We evaluate the limit distribution function of the free path length and of the input aimed parameter (the distance from the trajectory to the integer point we are interested in) for a given value of the output aimed parameter. This problem was studied earlier for a particle starting from the origin (the homogeneous case).

Keywords: analytic number theory, dynamical systems, continued fractions, Kloosterman sums, billiards, geometry of numbers.

Introduction

We introduce the following notation: $\|x\|$ is the distance from a real number x to the nearest integer, $\varphi(d)$ is the number of integers between 1 and d coprime to d (the Euler function), $\mu(d)$ is the Möbius function and $\delta_q(a) = 1$ if an integer a is divisible by q and $\delta_q(a) = 0$ otherwise (the function of divisibility by q).

We define the finite differences of a function $f(m,n)$ of two variables as follows:

$$\Delta_{1,0} f(m,n) = f(m+1,n) - f(m,n),$$
$$\Delta_{0,1} f(m,n) = f(m,n+1) - f(m,n),$$
$$\Delta_{1,1} f(m,n) = \Delta_{0,1}(\Delta_{1,0} f)(m,n) = \Delta_{1,0}(\Delta_{0,1} f)(m,n).$$

Let $f(x)$ and $g(x)$ be functions with the same domain and let $g(x) \geqslant 0$. Then the expressions

$$f(x) = O(g(x)), \qquad f(x) \ll g(x)$$

The research of the first author was financially supported by INTAS (grant no. 03-51-5070), by RBRF (grant no. 07-01-00306), and by the Far-Eastern Branch of the Russian Academy of Sciences (project no. 06-I-P-13-047).

The research of the second author was financially supported by INTAS (grant no. 03-51-5070), by RBRF (grant no. 07-01-00306), by the Far-Eastern Branch of the Russian Academy of Sciences (project no. 06-III-A-01-017), by the Russian Science Support Foundation, and by the Dynasty Foundation.

AMS 2000 *Mathematics Subject Classification.* Primary 82B20. Secondary 37D50, 37N20.

mean that the inequality $|f(x)| \leqslant cg(x)$ holds for some absolute positive constant c on the entire domain. If $c = c(\vartheta)$ (that is, the constant depends on some parameter ϑ), then we write

$$f(x) = O_\vartheta(g(x)), \qquad f(x) \underset{\vartheta}{\ll} g(x).$$

When studying rather fast-moving particles (in a crystal) whose trajectories are governed mainly by their multiple scattering on nuclei, we arrive at the following rather natural mathematical construction.

For fixed real numbers h and v in the intervals $\left(0, \frac{1}{8}\right)$ and $(-1, 1)$, respectively, the line (on the plane) defined parametrically by the rule

$$\big\{(-hv\sin\varphi + t\cos\varphi,\ hv\cos\varphi + t\sin\varphi) \in \mathbb{R}^2 \mid t \in (-\infty, +\infty)\big\} \tag{1}$$

and oriented in the direction $(\cos\varphi, \sin\varphi)$ passes at $t = 0$ through the point $O' = (-hv\sin\varphi, hv\cos\varphi)$, which is its nearest point to the origin $O = (0, 0)$ (O' is the projection of O on the line (1)). Another parametric representation

$$\big\{(x - t'\sin\varphi,\ y + t'\cos\varphi) \in \mathbb{R}^2 \mid t' \in (-\infty, +\infty)\big\} \tag{2}$$

defines the line perpendicular to (1) and passing through the point (x, y) at $t' = 0$. These lines meet at some point $M = M(\varphi)$ when

$$t = R(x, y) = x\cos\varphi + y\sin\varphi, \qquad t' = U(x, y) = x\sin\varphi - y\cos\varphi + hv.$$

Among the integer points (m, n) on the plane satisfying the conditions

$$R(m, n) > 0, \qquad |U(m, n)| < h,$$

we choose a point $(m(\varphi), n(\varphi))$ for which the quantity $R(m, n)$ takes the minimal value. Such a point $(m(\varphi), n(\varphi))$ always exists because, by Minkowski's theorem on linear forms, there is an integer pair $(m, n) \neq (0, 0)$ for which

$$|m\cos\varphi + n\sin\varphi| \leqslant (h(1 - |v|))^{-1}, \qquad |m\sin\varphi - n\cos\varphi| < h(1 - |v|).$$

Moreover,

$$|U(m, n)| = |m\sin\varphi - n\cos\varphi + hv| < h(1 - |v|) + h|v| = h.$$

By the *h-neighbourhood* of a point (x, y) we mean the open circle of radius h centred at (x, y). Then $(m(\varphi), n(\varphi))$ is an integer point $(m, n) \neq (0, 0)$ whose h-neighbourhood is intersected for the first time by a particle moving along the line (1) from the point O' in the positive direction. This implies the uniqueness of the pair $(m(\varphi), n(\varphi))$. We write

$$r(\varphi) = hR(m(\varphi), n(\varphi)), \qquad u(\varphi) = h^{-1}U(m(\varphi), n(\varphi)).$$

Here we have

$$0 < r(\varphi) < \frac{1}{1 - |v|}, \qquad -1 < u(\varphi) < 1.$$

Guided by the terminology of nuclear physics, we call the value $r = r(\varphi)$ the normalized *free path* and v and $u = u(\varphi)$ the normalized *output* and *input aimed parameters*.

Let

$$0 < r_0 < \frac{1}{1-|v|}\,, \qquad -1 < u_- < u_+ < 1,$$

and let $\chi_I(\,\cdot\,)$ be the characteristic function of the interval I on the line $(-\infty, +\infty)$. Our main result is the following theorem.

Theorem. *Let $|v| < c < 1$. Then the distribution function*

$$\Phi_v(h) = \Phi_v(h; \varphi_0, r_0, u_-, u_+) = \int_0^{\varphi_0} \chi_{[0,r_0]}(r(\varphi))\chi_{[u_-,u_+]}(u(\varphi))\,d\varphi$$

satisfies the following asymptotic formula for any $\varepsilon > 0$ as $h \to 0$:

$$\Phi_v(h) = \int_0^{\varphi_0}\int_0^{r_0}\int_{u_-}^{u_+} \rho(\varphi, r, v, u)\,d\varphi\,dr\,du + O_{\varepsilon,c}(h^{\frac{1}{2}-\varepsilon}),$$

which is uniform with respect to v, u_-, u_+ and $\varphi_0 \in [0, 2\pi]$ and has density

$$\rho(\varphi, r, v, u) = \rho(r, v, u) = \rho(r, u, v) = \rho(r, -u, -v)$$

of the following form for $u \geqslant |v|$:

$$\rho(r, u, v) = \begin{cases} \dfrac{6}{\pi^2}\,, & \text{if } \ 0 \leqslant r \leqslant \dfrac{1}{u+1}\,, \\ \dfrac{6}{\pi^2}\,\dfrac{1}{u-v}\left(\dfrac{1}{r} - 1 - v\right), & \text{if } \ \dfrac{1}{u+1} \leqslant r \leqslant \dfrac{1}{1+v}\,, \\ 0, & \text{if } \ \dfrac{1}{1+v} \leqslant r. \end{cases}$$

Remark 1. From a physical point of view, the function $\frac{1}{2\pi}\rho(\varphi, r, v, u)$ can be interpreted as the density of particles moving rectilinearly with unit speed at an angle φ after the first scattering with the output aimed parameter $V = hv$ in the h-neighbourhood of some node of the integer lattice and traversing a distance $R = h^{-1}r$ before the repeated scattering for the input aimed parameter hu.

Remark 2. The density $\rho(\varphi, r, v, u)$ does not depend on the angle φ (isotropy). Its symmetry under the replacement of (v, u) by (u, v), $(-u, -v)$ and $(-v, -u)$ is explained by the isotropy and the 'reversibility' of the particle trajectories.

Remark 3. Sinai proved [1] the ergodicity of a rectilinear billiard from which a circle of radius h is cut away. The statement of the problem on the asymptotic behaviour as $h \to 0$ of the distribution function for the length of the trajectory before the first collision with the deleted circle (collisions with the cushions are not taken into account) is also due to Sinai. This is the special case of the problem in question when $v = 0$, $u_- = 1$, $u_+ = 1$ and $\varphi_0 = 2\pi$.

Remark 4. For $v = 0$ (the homogeneous problem) the theorem was proved in [2]. In the simplest setting in Remark 3, an earlier result was obtained in [3] with a worse estimate for the remainder (of the form $O_\varepsilon(h^{\frac{1}{8}-\varepsilon})$).

Remark 5. It follows from the results of [4], which were proved by ergodic methods using Ratner's theorem on the classification of invariant ergodic measures under the action of unipotent flows, that the function $\Phi_v(h)$ has a limit as $h \to 0$ in the special case when $\varphi_0 = 2\pi$. This is not sufficient to prove the isotropy property.

Remark 6. The two-dimensional model treated in the present paper is of some interest in the theory of channelling for particles moving parallel to the crystallographic planes (see [5] and [6]).

§ 1. Properties of the integer pairs $(m(\varphi), n(\varphi))$

According to the definitions, we have

$$h^{-1}r\left(\varphi+\frac{\pi}{2}\right)=n\left(\varphi+\frac{\pi}{2}\right)\cos\varphi-m\left(\varphi+\frac{\pi}{2}\right)\sin\varphi,$$
$$hu\left(\varphi+\frac{\pi}{2}\right)=n\left(\varphi+\frac{\pi}{2}\right)\sin\varphi+m\left(\varphi+\frac{\pi}{2}\right)\cos\varphi+hv.$$

Since the set of integer points is mapped onto itself under rotation of the plane through an angle $\pi/2$ about the origin and orientation is preserved, it follows that the output aimed parameter v is also preserved, and

$$r\left(\varphi+\frac{\pi}{2}\right)=r(\varphi),\qquad u\left(\varphi+\frac{\pi}{2}\right)=u(\varphi).$$

Therefore,

$$m\left(\varphi+\frac{\pi}{2}\right)=-n(\varphi),\qquad n\left(\varphi+\frac{\pi}{2}\right)=m(\varphi).$$

Further,

$$h^{-1}r\left(\frac{\pi}{2}-\varphi\right)=n\left(\frac{\pi}{2}-\varphi\right)\cos\varphi+m\left(\frac{\pi}{2}-\varphi\right)\sin\varphi,$$
$$hu\left(\frac{\pi}{2}-\varphi\right)=-n\left(\frac{\pi}{2}-\varphi\right)\sin\varphi+m\left(\frac{\pi}{2}-\varphi\right)\cos\varphi+hv.$$

We consider reflection in the line $y = x$. In this case, the set of integer points on the plane is mapped onto itself but the orientation is reversed. Therefore, the output aimed parameter v is taken to $-v$, and

$$r\left(\frac{\pi}{2}-\varphi\right)=r(\varphi),\qquad u\left(\frac{\pi}{2}-\varphi\right)=-u(\varphi),$$
$$m\left(\frac{\pi}{2}-\varphi\right)=n(\varphi),\qquad n\left(\frac{\pi}{2}-\varphi\right)=m(\varphi).$$

Summarizing the above considerations and taking the equality $\rho(r, u, v) = \rho(r, -u, -v)$ into account, we can conclude that it is sufficient to prove the theorem in the case when $\varphi_0 \in (0, \pi/4)$.

We shall use another parametrization of the slope angle of the trajectory in terms of $\alpha = \alpha(\varphi) = \tan\varphi \in (0,1)$:

$$\sin\varphi = \frac{\alpha}{\sqrt{1+\alpha^2}}\,, \qquad \cos\varphi = \frac{1}{\sqrt{1+\alpha^2}}\,,$$
$$R(x,y) = \frac{x+\alpha y}{\sqrt{1+\alpha^2}}\,, \qquad U(x,y) = \frac{\alpha x - y}{\sqrt{1+\alpha^2}} + hv,$$
$$r(\varphi) = h\frac{m(\varphi)+\alpha n(\varphi)}{\sqrt{1+\alpha^2}}\,, \qquad u(\varphi) = \frac{\alpha m(\varphi) - n(\varphi)}{h\sqrt{1+\alpha^2}} + v.$$

Lemma 1. *The numbers $m(\varphi)$ and $n(\varphi)$ are coprime.*

Proof. Assume that $\mathrm{GCD}(m(\varphi), n(\varphi)) = q > 1$. Writing $m = m(\varphi)/q$ and $n = n(\varphi)/q$, we obtain

$$|U(m,n)| = \left|\frac{\alpha m - n}{\sqrt{1+\alpha^2}} + hv\right| = \left|\frac{1}{q}\,\frac{\alpha m(\varphi) - n(\varphi)}{\sqrt{1+\alpha^2}} + \frac{1}{q}\,hv + \frac{q-1}{q}\,hv\right|$$
$$= \left|\frac{1}{q}\,U(m(\varphi), n(\varphi)) + \frac{q-1}{q}\,hv\right| < \frac{1}{q}\,h + \frac{q-1}{q}\,h = h.$$

Here we have
$$R(m,n) = \frac{1}{q}\,R(m(\varphi), n(\varphi)) < R(m(\varphi), n(\varphi)),$$
which contradicts the definition of the pair $(m(\varphi), n(\varphi))$. Thus, our assumption is false and $q = 1$.

We note that the equation $(m(\varphi), n(\varphi)) = (1,0)$, $(m(\varphi), n(\varphi)) = (1,1)$ holds only for $\alpha \in (0, \vartheta_0)$, $\alpha \in (\vartheta_1, 1)$ respectively, where ϑ_0 and ϑ_1 are the roots of the equations (in α)

$$\frac{\alpha}{\sqrt{1+\alpha^2}} + hv = h, \qquad \frac{\alpha - 1}{\sqrt{1+\alpha^2}} + hv = -h$$

in the interval $(0,1)$. Here we have

$$0 < \vartheta_0 < \sqrt{8}\,h < 1 - \sqrt{8}\,h < \vartheta_1 < 1. \tag{3}$$

Since
$$m(\varphi)\sin\varphi - n(\varphi)\cos\varphi + hv \geqslant -h,$$

we have the inequalities

$$n(\varphi) \leqslant m(\varphi)\alpha + (1+v)h\sqrt{1+\alpha^2} \leqslant m(\varphi) + \frac{1}{2}\,.$$

Therefore, $n(\varphi) \leqslant m(\varphi)$.

Writing

$$\mathcal{M} = \big\{(m,n) \in \mathbb{N}^2 \mid 0 < n < m,\ \mathrm{GCD}(m,n) = 1\big\},$$

we can summarize what has been said as follows.

Remark 7. For any $\alpha \in [\vartheta_0, \vartheta_1]$ the pair $(m(\varphi), n(\varphi))$ belongs to the set $\mathcal{M}$.

Let $(m,n) \in \mathcal{M}$. We define positive integers m_+ and m_- by the conditions

$$nm_\pm \equiv \pm 1 \pmod{m}, \qquad 0 < m_\pm < m. \tag{4}$$

Since n and m are coprime and $m \geqslant 2$, it follows that m_+ and m_- are defined uniquely by their congruences with $+1$ and -1.

We also set

$$n_- = \frac{nm_- + 1}{m}, \qquad n_+ = \frac{nm_+ - 1}{m}. \tag{5}$$

Remark 8. Given (m, m_+), the number n is defined uniquely by the conditions $0 < n < m$ and $nm_+ \equiv 1 \pmod{m}$, and the number n_- can be recovered uniquely from (m,n). The same holds for the pair (m, m_-).

Lemma 2. *We have the following properties for the integers m_+, m_-, n_+ and n_- uniquely determined by a pair $(m,n) \in \mathcal{M}$:*

1) $0 \leqslant n_+ < m_+ < m$ *and* $1 \leqslant n_- \leqslant m_- < m$,
2) $(m_+, n_+) + (m_-, n_-) = (m,n)$,
3) $nm_+ - n_+m = n_-m - nm_- = n_-m_+ - n_+m_- = 1$.

Proof. The validity of 1) follows immediately from the equations (5). Adding the congruences in (4), we obtain

$$(m_+ + m_-)n \equiv 0 \pmod{m}.$$

Since n is coprime to m, it follows that $m_+ + m_- = km$ for some positive integer k. However, $m_+ + m_- < 2m$, and therefore $k = 1$. Adding the equalities in (5), we obtain the other relation, $n_+ + n_- = n$. Thus, the validity of 2) is proved. Finally, according to (5), we have

$$1 = \det\begin{pmatrix} n & n_+ \\ m & m_+ \end{pmatrix} = \det\begin{pmatrix} n_- & n \\ m_- & m \end{pmatrix} = \det\begin{pmatrix} n_- & n_+ + n_- \\ m_- & m_+ + m_- \end{pmatrix} = \det\begin{pmatrix} n_- & n_+ \\ m_- & m_+ \end{pmatrix},$$

as claimed in 3). The proof of the lemma is complete.

Lemma 3. *Let $\alpha = \alpha(\varphi) \in [\vartheta_0, \vartheta_1]$. A pair $(m,n) \in \mathcal{M}$ coincides with the pair $(m(\varphi), n(\varphi))$ if and only if*

$$U(m_+, n_+) \geqslant h, \qquad U(m_-, n_-) \leqslant -h, \qquad |U(m,n)| < h.$$

Proof. Assume that the numbers

$$U(m_+(\varphi), n_+(\varphi)), \qquad U(m_-(\varphi), n_-(\varphi)) \tag{6}$$

have the same sign. Then in accordance with Lemma 2, 2) we have

$$|U(m(\varphi),n(\varphi))| = \bigl|U(m_+(\varphi),n_+(\varphi)) + U(m_-(\varphi),n_-(\varphi)) - vh\bigr| > 2h - h = h,$$

which contradicts the definition of the pair $(m(\varphi),n(\varphi))$. Hence, our assumption is false and the numbers we are interested in have different signs.

We now assume that

$$U(m_+(\varphi),n_+(\varphi)) = \frac{\alpha m_+(\varphi) - n_+(\varphi)}{\sqrt{1+\alpha^2}} + vh \leqslant -h.$$

Then

$$\alpha m_+(\varphi) - n_+(\varphi) \leqslant -\sqrt{1+\alpha^2}\,(1+v)h < 0.$$

Since the sign of the number $U(m_-(\varphi),n_-(\varphi))$ is opposite to that of the number $U(m_+(\varphi),n_+(\varphi))$, it follows that

$$\alpha m_-(\varphi) - n_-(\varphi) \geqslant \sqrt{1+\alpha^2}\,(1-v)h > 0.$$

Then

$$\frac{n_-(\varphi)}{m_-(\varphi)} < \alpha < \frac{n_+(\varphi)}{m_+(\varphi)}\,.$$

However, this contradicts the equation

$$n_-(\varphi)m_+(\varphi) - n_+(\varphi)m_-(\varphi) = 1$$

in Lemma 2, and our assumption is false. Moreover, $|U(m_\pm(\varphi),n_\pm(\varphi))| \geqslant h$ by the definition of the pair $(m(\varphi),n(\varphi))$, that is, the first number in (6) is positive and the other negative. This proves the necessity of the conditions in Lemma 3.

Let us now prove their sufficiency. Suppose that there is an integer pair (m_1,n_1), $0 < m_1 < m$, such that $|U(m_1,n_1)| < h$. Having regard to the same considerations as those used in the proof of Lemma 1, we may assume that m_1 and n_1 are coprime. There are two coprime integers a and b for which the following relations hold (in accordance with Lemma 2, 3), the determinant of the system from which the numbers a and b are found is equal to 1):

$$am_+ + bm_- = m_1, \qquad an_+ + bn_- = n_1.$$

If one of the numbers a and b vanishes, then the other is equal to one, and the proposed inequality fails to hold (by assumption) for the pairs (m_+,n_+) and (m_-,n_-) obtained in this way. Therefore, a and b are non-zero. Suppose that $ab<0$. Then

$$\begin{aligned}|U(m,n)| &= \bigl|aU(m_+,n_+) + bU(m_-,n_-) + (1-a-b)vh\bigr| \\ &\geqslant |a|h + |b|h - |a+b-1|h \geqslant h.\end{aligned}$$

We have arrived at a contradiction, and therefore a and b are both positive. Then $m_1 = am_+ + bm_- \geqslant m_+ + m_- = m$, which again leads to a contradiction. Hence, the inequality $|U(m_1,n_1)| \geqslant h$ holds for all positive integers $m_1 < m$ and any integer n_1. This completes the proof of the lemma.

§ 2. Auxiliary transformations

Since the equation

$$\chi_{(u_-,u_+]}(u)=\chi_{(-v,u_+]}(u)-\chi_{(-v,u_-]}(u)$$

holds for $-v<u_-<u_+$ and the equation

$$\chi_{[u_-,u_+)}(u)=\chi_{[u_-,-v)}(u)-\chi_{[u_+,-v)}(u)$$

holds for $u_-<u_+<-v$, it suffices to prove the assertion of the theorem in the case when

$$-1<u_-\leqslant -v\leqslant u_+<1,$$

and we assume in what follows that these conditions are satisfied.

Let $(m,n)\in\mathcal{M}$. We denote by $I(m,n)=I(h,v,u_-,u_+;m,n)$ the subset of $[\vartheta_0,\vartheta_1]$ formed by all numbers α satisfying the conditions

$$(1-v)h\sqrt{1+\alpha^2}\leqslant \alpha m_+-n_+,\qquad \alpha m_--n_-\leqslant -(1+v)h\sqrt{1+\alpha^2}\,, \tag{7}$$

$$(u_--v)h\sqrt{1+\alpha^2}\leqslant \alpha m-n\leqslant (u_+-v)h\sqrt{1+\alpha^2}\,. \tag{8}$$

It follows immediately from (8) that

$$\left|\alpha-\frac{n}{m}\right|\leqslant 2\sqrt{2}\,\frac{h}{m} \tag{9}$$

for any $\alpha\in I(m,n)$.

We mentioned in the introduction that the pair $(m(\varphi),n(\varphi))$ is defined uniquely by the angle φ. Therefore, the domain of integration with respect to α (after replacing φ by $\arctan\alpha$) in the integral defining $\Phi_v(h)$ is partitioned, according to Lemma 3, into disjoint closed intervals $I(m,n)$, $(m,n)\in\mathcal{M}$, $[0,\vartheta_0]$, and $[\vartheta_1,1]$. Estimating the integrals over the last two intervals (using the inequalities in (3)) and setting $\alpha_0=\tan\varphi_0$, we obtain

$$\Phi_v(h)=\sum_{(m,n)\in\mathcal{M}}\int_{I(m,n)}\chi_{[0,\alpha_0]}(\alpha)\chi_{[0,r_0]}\biggl(h\,\frac{m+\alpha n}{\sqrt{1+\alpha^2}}\biggr)\frac{d\alpha}{1+\alpha^2}+O(h).$$

Let

$$\mathcal{M}(R)=\Bigl\{(m,n)\in\mathcal{M}\mid \sqrt{m^2+n^2}\leqslant R\Bigr\},\quad \mathcal{M}_t(R)=\biggl\{(m,n)\in\mathcal{M}(R)\biggm|\frac{n}{m}\leqslant t\biggr\},$$

for $R\in[1,\infty)$ and $t\in(0,1)$.

Lemma 4. *The following equation holds for* $r_0<(1-|v|)^{-1}$ *and* $|v|\leqslant c<1$:

$$\Phi_v(h)=\sum_{(m,n)\in\mathcal{M}_{\alpha_0}(r_0h^{-1})}\operatorname{mes}(I(m,n))\biggl(1+\Bigl(\frac{n}{m}\Bigr)^2\biggr)^{-1}+O_c(h\ln(h^{-1})).$$

Proof. Since

$$\left(\frac{m+\alpha n}{\sqrt{1+\alpha^2}}\right)^2+\left(\frac{\alpha m-n}{\sqrt{1+\alpha^2}}\right)^2=m^2+n^2,$$

it follows that

$$m^2+n^2-(2h)^2<\left(\frac{m+\alpha n}{\sqrt{1+\alpha^2}}\right)^2\leqslant m^2+n^2$$

for $\alpha\in I(m,n)$ (see (8)). Hence, by (9),

$$\Phi_v(h)-\sum_{(m,n)\in\mathcal{M}(r_0h^{-1})}\int_{I(m,n)}\chi_{[0,\alpha_0]}(\alpha)\frac{d\alpha}{1+\alpha^2}\ll h+\sum_{\substack{(m,n)\in\mathcal{M}\\ 0\leqslant m^2+n^2-(r_0h^{-1})^2\leqslant(2h)^2}}\int_{I(m,n)}d\alpha$$

$$\ll\sum_{1\leqslant m\leqslant r_0h^{-1}}\sum_{0\leqslant n^2-((r_0h^{-1})^2-m^2)\leqslant\frac12}\frac{h}{m}\ll h\sum_{1\leqslant m\leqslant r_0h^{-1}}\frac{1}{m}\underset{c}{\ll}h\ln(h^{-1}).$$

If

$$\chi_{[0,\alpha_0]}(\alpha)\neq\chi_{[0,\alpha_0]}\left(\frac{n}{m}\right)$$

for some $\alpha\in I(m,n)$, then it follows from the inequality (9) that

$$\left|\alpha_0-\frac{n}{m}\right|\leqslant 2\sqrt{2}\,\frac{h}{m}\quad\Longrightarrow\quad|\alpha_0m-n|\leqslant 2\sqrt{2}\,h<\frac12\,.$$

Therefore,

$$\begin{aligned}&\Phi_v(h)-\sum_{(m,n)\in\mathcal{M}_{\alpha_0}(r_0h^{-1})}\int_{I(m,n)}\frac{d\alpha}{\sqrt{1+\alpha^2}}\\&\quad\underset{c}{\ll}h\ln(h^{-1})+\sum_{\substack{(m,n)\in\mathcal{M}_{\alpha_0}(r_0h^{-1})\\|\alpha_0m-n|<1/2}}\operatorname{mes}(I(m,n))\\&\quad\ll h\ln(h^{-1})+\sum_{m\leqslant 1+r_0h^{-1}}\frac{h}{m}\underset{c}{\ll}h\ln(h^{-1}).\end{aligned}$$

According to (9), for any $\alpha\in I(m,n)$ we have

$$(1+\alpha^2)^{-1}-\left(1+\left(\frac{n}{m}\right)^2\right)^{-1}\ll\frac{h}{m}\,,$$

and we finally see that

$$\begin{aligned}&\Phi_v(h)-\sum_{(m,n)\in\mathcal{M}_{\alpha_0}(r_0h^{-1})}\operatorname{mes}(I(m,n))\left(1+\left(\frac{n}{m}\right)^2\right)^{-1}\\&\quad\underset{c}{\ll}h\ln(h^{-1})+\sum_{1\leqslant m<n\leqslant r_0h^{-1}}\frac{h}{m}\,\frac{h}{m}\underset{c}{\ll}h\ln(h^{-1}).\end{aligned}$$

This completes the proof of the lemma.

We write $\alpha = \frac{n}{m} + \beta$ and, using the relations in Lemma 2, 3), represent the inequalities (7) and (8) in the form

$$-\frac{1}{mm_+} + (1-v)\frac{h}{m_+}\sqrt{1+\left(\frac{n}{m}+\beta\right)^2} \leqslant \beta \leqslant \frac{1}{mm_-} - (1+v)\frac{h}{m_-}\sqrt{1+\left(\frac{n}{m}+\beta\right)^2},$$

$$(u_- - v)\frac{h}{m}\sqrt{1+\left(\frac{n}{m}+\beta\right)^2} \leqslant \beta \leqslant (u_+ - v)\frac{h}{m}\sqrt{1+\left(\frac{n}{m}+\beta\right)^2}.$$

Let

$$f_1(\beta) = \beta + \frac{1}{mm_+} - (1-v)\frac{h}{m_+}\sqrt{1+\left(\frac{n}{m}+\beta\right)^2}.$$

For an arbitrary $f_1'(\beta)$ we see that

$$f_1'(\beta) = 1 - \frac{(1-v)h(\frac{n}{m}+\beta)}{m_+\sqrt{1+(\frac{n}{m}+\beta)^2}} \geqslant 1-(1-v)\frac{h}{m_+} \geqslant 1-2h > \frac{1}{2}.$$

Therefore, $f_1(\beta)$ is an increasing function, and the equation $f_1(\beta) = 0$ has a unique root, which we denote by $\lambda_-(m,n)$. One can prove in the same way that the functions

$$f_2(\beta) = \beta - \frac{1}{mm_-} + (1+v)\frac{h}{m_-}\sqrt{1+\left(\frac{n}{m}+\beta\right)^2},$$

$$f_3(\beta) = \beta - (u_- - v)\frac{h}{m}\sqrt{1+\left(\frac{n}{m}+\beta\right)^2},$$

$$f_4(\beta) = \beta - (u_+ - v)\frac{h}{m}\sqrt{1+\left(\frac{n}{m}+\beta\right)^2}$$

are increasing and change sign on the interval in question. For this reason, these functions vanish at unique points, say, $\lambda_+(m,n)$, $\gamma_-(m,n)$, $\gamma_+(m,n)$, respectively. Hence, the conditions imposed on β can be represented in the form

$$\lambda_-(m,n) \leqslant \beta \leqslant \lambda_+(m,n), \qquad \gamma_-(m,n) \leqslant \beta \leqslant \gamma_+(m,n).$$

We set

$$\widetilde{\lambda}_-(m,n) = -\frac{1}{mm_+} + (1-v)\frac{h}{m_+}\sqrt{1+\left(\frac{n}{m}\right)^2},$$

$$\widetilde{\lambda}_+(m,n) = \frac{1}{mm_-} - (1+v)\frac{h}{m_-}\sqrt{1+\left(\frac{n}{m}\right)^2},$$

$$\widetilde{\gamma}_-(m,n) = (u_- - v)\frac{h}{m}\sqrt{1+\left(\frac{n}{m}\right)^2}, \qquad \widetilde{\gamma}_+(m,n) = (u_+ - v)\frac{h}{m}\sqrt{1+\left(\frac{n}{m}\right)^2}.$$

Since

$$0 = f_1(\lambda_-(m,n)) = \lambda_-(m,n) - \widetilde{\lambda}_-(m,n) \\ + (1-v)\frac{h}{m_+}\Biggl(\sqrt{1+\Bigl(\frac{n}{m}\Bigr)^2} - \sqrt{1+\Bigl(\frac{n}{m}+\lambda_-(m,n)\Bigr)^2}\,\Biggr),$$

we see from Lagrange's theorem on the difference between two values of a function (using the inequality (9)) that

$$\lambda_-(m,n) - \widetilde{\lambda}_-(m,n) \ll \frac{h}{m_+}\,\lambda_-(m,n) \ll \frac{h^2}{mm_+}\,.$$

The following three bounds are obtained in just the same way:

$$\lambda_+(m,n) - \widetilde{\lambda}_+(m,n) \ll \frac{h^2}{mm_-}\,, \qquad \gamma_\pm(m,n) - \widetilde{\gamma}_\pm(m,n) \ll \frac{h^2}{m^2}\,.$$

Therefore,

$$\operatorname{mes}(I(m,n)) = \operatorname{mes}(J(m,n)) + O\biggl(\frac{h^2}{mm_-} + \frac{h^2}{mm_+}\biggr), \tag{10}$$

where $J(m,n)$ is the set formed by all numbers β satisfying the condition

$$\max\bigl\{\widetilde{\lambda}_-(m,n), \widetilde{\gamma}_-(m,n)\bigr\} \leqslant \beta \leqslant \min\bigl\{\widetilde{\lambda}_+(m,n), \widetilde{\gamma}_+(m,n)\bigr\}.$$

Since $(u_- - v) \leqslant (u_+ - v)$, it follows that $\widetilde{\gamma}_-(m,n) \leqslant \widetilde{\gamma}_+(m,n)$. Hence, $J(m,n)$ is non-empty only for the pairs $(m,n) \in \mathcal{M}$ for which the inequalities

$$\widetilde{\lambda}_-(m,n) \leqslant \widetilde{\lambda}_+(m,n), \qquad \widetilde{\lambda}_-(m,n) \leqslant \widetilde{\gamma}_+(m,n), \qquad \widetilde{\gamma}_-(m,n) \leqslant \widetilde{\lambda}_+(m,n)$$

hold simultaneously. Using the relation $m_+ + m_- = m$, we write these inequalities in the form

$$w(m,n) = \bigl((1+v)m_+ + (1-v)m_-\bigr)h\sqrt{1+\Bigl(\frac{n}{m}\Bigr)^2} \leqslant 1, \tag{11}$$

$$\bigl((1-u_+)m_+ + (1-v)m_-\bigr)h\sqrt{1+\Bigl(\frac{n}{m}\Bigr)^2} \leqslant 1, \tag{12}$$

$$\bigl((1+v)m_+ + (1+u_-)m_-\bigr)h\sqrt{1+\Bigl(\frac{n}{m}\Bigr)^2} \leqslant 1. \tag{13}$$

Since $1-u_+ \leqslant 1+v$ and $1+u_- \leqslant 1-v$ by assumption, the inequalities (12) and (13) are consequences of (11), and we have the following remark.

Remark 9. The closed interval $J(m,n)$, $(m,n) \in \mathcal{M}$, is non-empty if and only if $w(m,n) \leqslant 1$.

It can readily be seen that (11) is equivalent to the inequality

$$\widetilde{\lambda}_-(m,n) \leqslant \widetilde{\gamma}_-(m,n) \quad \text{for} \ \ u_- = -v$$

and to the inequality

$$\widetilde{\lambda}_+(m,n) \geqslant \widetilde{\gamma}_+(m,n) \quad \text{for} \ \ u_+ = -v.$$

Therefore, the interval $J(m,n)$ is divided by the point

$$\beta_0 = -2v\frac{h}{m}\sqrt{1+\left(\frac{n}{m}\right)^2}$$

into two intervals $J_+(m,n)$ and $J_-(m,n)$ which are obtained from $J(m,n)$ by replacing the pair (u_-,u_+) by $(-v,u_+)$ and $(u_-,-v)$, respectively. Here

$$\begin{gathered}
J_+(m,n) = \bigl\{\beta \in \mathbb{R} \bigm| \beta_0 \leqslant \beta \leqslant \min\{\lambda_+(m,n),\gamma_+(m,n)\}\bigr\}, \\
J_-(m,n) = \bigl\{\beta \in \mathbb{R} \bigm| \max\{\lambda_-(m,n),\gamma_-(m,n)\} \leqslant \beta \leqslant \beta_0\bigr\}, \\
\operatorname{mes}(J(m,n)) = \operatorname{mes}(J_+(m,n)) + \operatorname{mes}(J_-(m,n)).
\end{gathered}$$

Lemma 5. *Under the assumptions of Lemma 4,*

$$\Phi_v(h) = \Psi_v^+(h) + \Psi_v^-(h) + O_c(h\ln(h^{-1})),$$

where

$$\Psi_v^\pm(h) = \sum_{\substack{(m,n)\in\mathcal{M}_{\alpha_0}(r_0h^{-1}) \\ w(m,n)\leqslant 1}} \operatorname{mes}(J_\pm(m,n))\left(1+\left(\frac{n}{m}\right)^2\right)^{-1}.$$

Proof. Applying Lemma 4 and taking the asymptotic equality (10) into account, we obtain (see also Remark 8)

$$\begin{aligned}
\Phi_v(h) - \Psi_v^+(h) - \Psi_v^-(h) &\ll_c \sum_{(m,n)\in\mathcal{M}(r_0h^{-1})} \left(\frac{h^2}{mm_+} + \frac{h^2}{mm_-}\right) + h\ln(h^{-1}) \\
&\ll \sum_{0<m'<m\leqslant r_0h^{-1}} \frac{h^2}{m'm} + h\ln(h^{-1}) \ll_c h\ln(h^{-1}).
\end{aligned}$$

§ 3. Application of estimates for Kloosterman sums

In accordance with the definitions,

$$\begin{aligned}
\frac{\operatorname{mes}(J_+(m,n))}{1+\left(\frac{n}{m}\right)^2} &= \frac{2v\frac{h}{m}\sqrt{1+\left(\frac{n}{m}\right)^2} + \min\{\lambda_+(m,n),\gamma_+(m,n)\}}{1+\left(\frac{n}{m}\right)^2} \\
&= \frac{h}{m}\,g_+\left(\frac{n}{m},\,\frac{m_-}{m}\right),
\end{aligned}$$

where

$$g_+(x,y) = \frac{2v + \min\{u_+ - v, s_+(x,y)\}}{\sqrt{1+x^2}}\,,$$
$$s_+(x,y) = \frac{1}{y}\biggl(\frac{1}{mh\sqrt{1+x^2}} - (1+v)\biggr).$$

Here

$$w(m,n) \geqslant 1 \quad\Longleftrightarrow\quad s_+\biggl(\frac{n}{m}\,,\ \frac{m_-}{m}\biggr) \geqslant -2v.$$

Similarly,

$$\frac{\operatorname{mes}(J_-(m,n))}{1+\bigl(\frac{n}{m}\bigr)^2} = \frac{-2v\frac{h}{m}\sqrt{1+\bigl(\frac{n}{m}\bigr)^2} - \max\{\lambda_-(m,n), \gamma_-(m,n)\}}{1+\bigl(\frac{n}{m}\bigr)^2}$$
$$= \frac{h}{m}\, g_-\biggl(\frac{n}{m}\,,\ \frac{m_+}{m}\biggr),$$

where

$$g_-(x,y) = \frac{-2v + \min\{v - u_-, s_-(x,y)\}}{\sqrt{1+x^2}}\,,$$
$$s_-(x,y) = \frac{1}{y}\biggl(\frac{1}{mh\sqrt{1+x^2}} - (1-v)\biggr).$$

Here

$$w(m,n) \geqslant 1 \quad\Longleftrightarrow\quad s_-\biggl(\frac{n}{m}\,,\ \frac{m_+}{m}\biggr) \geqslant 2v.$$

We also note that the condition $\sqrt{m^2+n^2} \leqslant r_0 h^{-1}$ can be written in the form

$$\sqrt{1+\biggl(\frac{n}{m}\biggr)^2} \leqslant \frac{r_0}{mh}\,.$$

Let $\alpha_1 = \min\{\alpha_0, \sqrt{r_0^2(mh)^{-2}-1}\,\}$. We define a function $f_\pm$ on the rectangle $[0,\alpha_1]\times[0,1]$ by setting

$$f_\pm(x,y) = \begin{cases} g_\pm(x,y), & \text{if } \mp 2v \leqslant s_\pm(x,y), \\ 0, & \text{if } \mp 2v > s_\pm(x,y). \end{cases}$$

Then

$$\Psi_v^\pm(h) = h \sum_{1<m\leqslant r_0 h^{-1}} \frac{W_\pm(m)}{m}\,,$$

where

$$W_\pm(m) = \sum_{\substack{1\leqslant n\leqslant m' \\ 1\leqslant n'\leqslant m}} \delta_m(nn'\pm 1) f_\pm\biggl(\frac{n}{m}\,,\ \frac{n'}{m}\biggr)$$

for $m' = [\alpha_1 m]$.

Applying the Abel transformation

$$\sum_{0<l\leqslant N} a(l)\, b(l) = a(N) \sum_{0<k\leqslant N} b(k) - \sum_{0<l<N} (a(l+1)-a(l))\Biggl(\sum_{0<k\leqslant l} b(k)\Biggr)$$

to the second variable, we obtain

$$W_{\pm}(m) = W_{\pm}^{(0)}(m) - W_{\pm}^{(1)}(m),$$

where

$$W_{\pm}^{(0)}(m) = \sum_{1\leqslant n\leqslant m'} f_{\pm}\biggl(\frac{n}{m},\frac{m}{m}\biggr)\Biggl(\sum_{1\leqslant n'\leqslant m} \delta_m(nn'\pm 1)\Biggr),$$

$$W_{\pm}^{(1)}(m) = \sum_{\substack{1\leqslant n\leqslant m'\\ 1\leqslant k'<m}} \Delta_{0,1} f_{\pm}\biggl(\frac{n}{m},\frac{k'}{m}\biggr)\Biggl(\sum_{1\leqslant n'\leqslant k'} \delta_m(nn'\pm 1)\Biggr).$$

Again making the Abel transformation in both the sums with respect to the first variable, we see that

$$W_{\pm}^{(0)}(m) = W_{\pm}^{(0,0)}(m) - W_{\pm}^{(0,1)}(m),$$
$$W_{\pm}^{(1)}(m) = W_{\pm}^{(1,0)}(m) - W_{\pm}^{(1,1)}(m),$$

where

$$W_{\pm}^{(0,0)}(m) = f_{\pm}\biggl(\frac{m'}{m},\frac{m}{m}\biggr)\sum_{\substack{1\leqslant n\leqslant m'\\ 1\leqslant n'\leqslant m}} \delta_m(nn'\pm 1),$$

$$W_{\pm}^{(0,1)}(m) = \sum_{1\leqslant k<m'} \Delta_{1,0} f_{\pm}\biggl(\frac{k}{m},\frac{m}{m}\biggr)\Biggl(\sum_{\substack{1\leqslant n\leqslant k\\ 1\leqslant n'\leqslant m}} \delta_m(nn'\pm 1)\Biggr),$$

$$W_{\pm}^{(1,0)}(m) = \sum_{1\leqslant k'<m} \Delta_{0,1} f_{\pm}\biggl(\frac{m'}{m},\frac{k'}{m}\biggr)\Biggl(\sum_{\substack{1\leqslant n\leqslant m'\\ 1\leqslant n'\leqslant k'}} \delta_m(nn'\pm 1)\Biggr),$$

$$W_{\pm}^{(1,1)}(m) = \sum_{\substack{1\leqslant k<m'\\ 1\leqslant k'<m}} \Delta_{1,1} f_{\pm}\biggl(\frac{k}{m},\frac{k'}{m}\biggr)\Biggl(\sum_{\substack{1\leqslant n\leqslant k\\ 1\leqslant n'\leqslant k'}} \delta_m(nn'\pm 1)\Biggr).$$

We now apply the asymptotic equality ($\forall\,\varepsilon>0$)

$$\sum_{\substack{1\leqslant n\leqslant k\\ 1\leqslant n'\leqslant k'}} \delta_m(nn'\pm 1) = \frac{\varphi(m)}{m^2}\,kk' + O_\varepsilon(m^{\frac12+\varepsilon})$$

for $1\leqslant k,k'<m$. This inequality can be proved in a standard way (see, for instance, [7]) using Estermann's estimates for Kloosterman sums [8]. As a result, we obtain the eight equalities ($0\leqslant i,j\leqslant 1$)

$$W_{\pm}^{(i,j)}(m) = \frac{\varphi(m)}{m^2}\,D_{\pm}^{(i,j)}(m) + O_\varepsilon(G_{\pm}^{(i,j)}(m)m^{\frac12+\varepsilon}),$$

where

$$D_\pm^{(0,0)}(m) = f_\pm\left(\frac{m'}{m}, \frac{m}{m}\right)m'm, \qquad G_\pm^{(0,0)}(m) = 1,$$

$$D_\pm^{(0,1)}(m) = \sum_{1\leqslant k<m'} \Delta_{1,0}\, f_\pm\left(\frac{k}{m}, \frac{m}{m}\right)km,$$

$$G_\pm^{(0,1)}(m) = \sum_{1\leqslant k<m'} \left|\Delta_{1,0}\, f_\pm\left(\frac{k}{m}, \frac{m}{m}\right)\right|,$$

$$D_\pm^{(1,0)}(m) = \sum_{1\leqslant k'<m} \Delta_{0,1}\, f_\pm\left(\frac{m'}{m}, \frac{k'}{m}\right)m'k',$$

$$G_\pm^{(1,0)}(m) = \sum_{1\leqslant k'<m} \left|\Delta_{0,1}\, f_\pm\left(\frac{m'}{m}, \frac{k'}{m}\right)\right|,$$

$$D_\pm^{(1,1)}(m) = \sum_{\substack{1\leqslant k<m'\\1\leqslant k'<m}} \Delta_{1,1} f_\pm\left(\frac{k}{m}, \frac{k'}{m}\right)kk',$$

$$G_\pm^{(1,1)}(m) = \sum_{\substack{1\leqslant k<m'\\1\leqslant k'<m}} \left|\Delta_{1,1}\, f_\pm\left(\frac{k}{m}, \frac{k'}{m}\right)\right|.$$

It can readily be seen that the Abel transformation with respect to two variables, when applied to the sum

$$\sum_{\substack{1\leqslant n\leqslant m'\\1\leqslant n'\leqslant m}} f_\pm\left(\frac{n}{m}, \frac{n'}{m}\right)b(n,n')$$

with $b(n,n')=1$, leads to the equation

$$S_\pm(m) = \sum_{\substack{1\leqslant n\leqslant m'\\1\leqslant n'\leqslant m}} f_\pm\left(\frac{n}{m}, \frac{n'}{m}\right) = D_\pm^{(0,0)}(m) - D_\pm^{(0,1)}(m) - D_\pm^{(1,0)}(m) + D_\pm^{(1,1)}(m).$$

Therefore,

$$W_\pm(m) = \frac{\varphi(m)}{m^2}\, S_\pm(m) + O_\varepsilon(G_\pm(m)m^{\frac{1}{2}+\varepsilon}) \tag{14}$$

for any $\varepsilon>0$, where

$$G_\pm(m) = G_\pm^{(0,0)}(m) + G_\pm^{(0,1)}(m) + G_\pm^{(1,0)}(m) + G_\pm^{(1,1)}(m).$$

We shall need the following obvious remarks.

Remark 10. Both of the functions $f_\pm(x,\,\cdot\,)$ are monotone with respect to the second variable y for any fixed x.

Remark 11. Both of the functions $f_\pm(\,\cdot\,,y)$ are continuous with respect to the first variable on the interval $[0,\alpha_1]$ for any fixed y. Moreover, these functions are continuously differentiable, except possibly for a single point, and the inequality

$$\Delta_{1,0}\, f_\pm\left(\frac{k}{m},\frac{k'}{m}\right) \ll \frac{1}{m}$$

holds uniformly with respect to (x,y).

Lemma 6. *For any positive integer $m \geqslant 2$ we have $G_\pm(m) \ll 1$.*

Proof. It suffices to show that

$$G_\pm^{(i,j)}(m) \ll 1$$

for $0 \leqslant i,j \leqslant 1$. When $i=j=0$ we have $G_\pm^{(0,0)}(m)=1$, and the desired inequality holds. By Remark 10, when $i=1$ and $j=0$ we see that

$$G_\pm^{(1,0)}(m) = -\sum_{1\leqslant k'<m} \Delta_{0,1}\, f_\pm\left(\frac{m'}{m},\frac{k'}{m}\right) = f_\pm\left(\frac{m'}{m},\frac{1}{m}\right) - f_\pm\left(\frac{m'}{m},\frac{m}{m}\right) \ll 1.$$

By Remark 11, when $i=0$ and $j=1$ we obtain

$$G_\pm^{(0,1)}(m) \ll \sum_{1\leqslant k<m'} \frac{1}{m} \ll 1.$$

It remains to treat the most complicated case, $i=j=1$.

We note that

$$\frac{\partial^2 g_\pm}{\partial x\,\partial y}(x,y) = \begin{cases} g''_\pm(x,y), & \text{if } s_\pm(x,y) < \pm(u_\pm - v),\\ 0, & \text{if } s_\pm(x,y) > \pm(u_\pm - v), \end{cases}$$

on the rectangle $[0,m'/m]\times[0,1]$, where

$$g''_\pm(x,y) = \frac{x}{mhy^2(1+x^2)^2}\Bigl(2-(1\pm v)mh\sqrt{1+x^2}\,\Bigr).$$

Let $\alpha_\pm$ be the positive roots of the equations

$$2-(1\pm v)mh\sqrt{1+x^2} = 0$$

in x.

Suppose that $m\alpha_\pm \geqslant m'$. Then except at points on the curves

$$s_\pm(x,y) = \pm(u_\pm - v), \tag{15}$$

the mixed derivative of the function $g_\pm$ is non-negative on the rectangle under consideration. Hence,

$$\Delta_{1,1}\, f_\pm\left(\frac{k}{m},\frac{k'}{m}\right) \geqslant 0$$

at all points $\left(\frac{k}{m}, \frac{k'}{m}\right)$ with $1 \leqslant k < m'$ and $1 \leqslant k' < m$, except possibly at the points at which the curves in (15) intersect the square $\left[\frac{k}{m}, \frac{k+1}{m}\right] \times \left[\frac{k'}{m}, \frac{k'+1}{m}\right]$. The number of these points is $O(m)$ and, according to Remark 11, at these points we have

$$\left|\Delta_{1,1} f_{\pm}\left(\frac{k}{m}, \frac{k'}{m}\right)\right| \leqslant \left|\Delta_{1,0} f_{\pm}\left(\frac{k}{m}, \frac{k'}{m}\right)\right| + \left|\Delta_{1,0} f_{\pm}\left(\frac{k}{m}, \frac{k'+1}{m}\right)\right| \ll \frac{1}{m}.$$

Therefore,

$$\begin{aligned} G_{\pm}^{(1,1)}(m) &= \sum_{\substack{1\leqslant k<m' \\ 1\leqslant k'<m}} \Delta_{1,1} f_{\pm}\left(\frac{k}{m}, \frac{k'}{m}\right) \\ &= \sum_{k=1}^{m'-1}\left(\Delta_{1,0} f_{\pm}\left(\frac{k}{m}, \frac{m}{m}\right) - \Delta_{1,0} f_{\pm}\left(\frac{k}{m}, \frac{1}{m}\right)\right) \\ &= f_{\pm}\left(\frac{m'}{m}, \frac{m}{m}\right) - f_{\pm}\left(\frac{1}{m}, \frac{m}{m}\right) - f_{\pm}\left(\frac{m'}{m}, \frac{1}{m}\right) + f_{\pm}\left(\frac{1}{m}, \frac{1}{m}\right) \ll 1. \end{aligned}$$

Now suppose that $m\alpha_{\pm} < m'$. We denote by $l_{\pm}$ the largest integers not exceeding $m\alpha_{\pm}$ and decompose the sum $G_{\pm}^{(1,1)}(m)$ into three sums, $G'_{\pm}$, $G''_{\pm}$ and $G'''_{\pm}$. We put the summands with $0 < k' < l_{\pm}$ into the first sum, those with $k' = l_{\pm}$ into the second, and those with $l_{\pm} < k' < m'$ into the third. The sums $G'_{\pm}$ and $G'''_{\pm}$ can be estimated using the same lines of reasoning as in the previous case ($m\alpha_{\pm} \geqslant m'$) because, in the first case, the mixed derivative is everywhere non-negative and, in the second, this derivative is everywhere non-positive (except at points on the curve (15)). Moreover, paying heed to Remark 10, we have

$$\begin{aligned} G''_{\pm} &= \sum_{1\leqslant k'<m}\left|\Delta_{1,1} f_{\pm}\left(\frac{l_{\pm}}{m}, \frac{k'}{m}\right)\right| \\ &\leqslant \sum_{1\leqslant k'<m}\left(\left|\Delta_{0,1} f_{\pm}\left(\frac{l_{\pm}}{m}, \frac{k'}{m}\right)\right| + \left|\Delta_{0,1} f_{\pm}\left(\frac{1+l_{\pm}}{m}, \frac{k'}{m}\right)\right|\right) \\ &= f_{\pm}\left(\frac{l_{\pm}}{m}, \frac{1}{m}\right) - f_{\pm}\left(\frac{l_{\pm}}{m}, \frac{m}{m}\right) + f_{\pm}\left(\frac{1+l_{\pm}}{m}, \frac{1}{m}\right) - f_{\pm}\left(\frac{1+l_{\pm}}{m}, \frac{m}{m}\right) \ll 1. \end{aligned}$$

The proof of the lemma is complete.

§ 4. Distinguishing the leading term

Let $F(x)$ be an arbitrary fixed piecewise-differentiable function on the interval $[x_0, x_1]$ with bounded derivative. As is well known,

$$\sum_{y_0 \leqslant \frac{k}{N} \leqslant y_2} F\left(\frac{k}{N}\right) = N\int_{y_0}^{y_1} F(x)\,dx + O(1) \tag{16}$$

for $x_0 \leqslant y_0 < y_1 \leqslant x_1$. Applying the asymptotic equality (16) twice and taking Remarks 10 and 11 into account, we obtain

$$\begin{aligned} S_\pm(m) &= \sum_{1\leqslant n'\leqslant m}\biggl(m\int_0^{\alpha_1} f_\pm\Bigl(x,\frac{n'}{m}\Bigr)\,dx + O(1)\biggr) \\ &= m^2\int_0^{\alpha_1}\int_0^1 f_\pm(x,y)\,dx\,dy + O(m). \end{aligned}$$

Applying Lemmas 5 and 6 together with the asymptotic equality (14), we see that

$$\Phi_v(h) = h\int_0^{\alpha_0}\int_0^1 (Q_+(x,y)+Q_-(x,y))\frac{dx\,dy}{\sqrt{1+x^2}} + O_{c,\varepsilon}(h^{\frac12-\varepsilon})$$

for any $\varepsilon>0$, where

$$Q_\pm(x,y) = \sum_{mh\sqrt{1+x^2}\leqslant r_0}\frac{\varphi(m)}{m}\,\Theta_\pm\Bigl(y, mh\sqrt{1+x^2}\Bigr),$$

$$\Theta_+(y,r) = \chi_{[r,\infty)}\biggl(\frac{1}{1+v(1-2y)}\biggr)\biggl(2v+\min\biggl\{u_+-v,\ \frac{1}{y}\Bigl(\frac{1}{r}-(1+v)\Bigr)\biggr\}\biggr),$$

$$\Theta_-(y,r) = \chi_{[r,\infty)}\biggl(\frac{1}{1-v(1-2y)}\biggr)\biggl(-2v+\min\biggl\{v-u_-,\ \frac{1}{y}\Bigl(\frac{1}{r}-(1-v)\Bigr)\biggr\}\biggr).$$

The first factors in the formulae for $\Theta_+(y,r)$ and $\Theta_-(y,r)$ ensure the validity of the conditions

$$s_+(x,y)\geqslant -2v, \qquad s_-(x,y)\geqslant 2v$$

mentioned at the beginning of § 3.

Since

$$\frac{\varphi(m)}{m} = \sum_{d\mid m}\frac{\mu(d)}{d} = \sum_{dn=m}\frac{\mu(d)}{d}\,,$$

it follows that

$$Q_\pm(x,y) = \sum_{dh\sqrt{1+x^2}\leqslant r_0}\frac{\mu(d)}{d}\sum_{ndh\sqrt{1+x^2}\leqslant r_0}\Theta_\pm\Bigl(y, ndh\sqrt{1+x^2}\Bigr).$$

Since $ndh\sqrt{1+x^2}\leqslant r_0\leqslant\frac{1}{1-c}$ and the function $\Theta_\pm(y,r)$ is bounded and monotone with respect to r, we can see by replacing the inner sum over n by the corresponding integral that

$$\begin{aligned} Q_\pm(x,y) &= \sum_{dh\sqrt{1+x^2}\leqslant r_0}\frac{\mu(d)}{d}\biggl(\frac{1}{dh\sqrt{1+x^2}}\int_0^{r_0}\Theta_\pm(y,r)\,dr + O_c(1)\biggr) \\ &= \frac{1}{h\sqrt{1+x^2}}\biggl(\sum_{dh\sqrt{1+x^2}\leqslant r_0}\frac{\mu(d)}{d^2}\biggr)\biggl(\int_0^{r_0}\Theta_\pm(y,r)\,dr\biggr) + O_c(\ln(h^{-1})). \end{aligned}$$

Since

$$\sum_{dh\sqrt{1+x^2}\leqslant r_0} \frac{\mu(d)}{d^2} = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} + O_c(h^{-1}) = \frac{1}{\zeta(2)} + O_c(h^{-1}) = \frac{6}{\pi^2} + O_c(h^{-1}),$$

it follows that

$$Q_{\pm}(x,y) = \frac{6}{\pi^2 h\sqrt{1+x^2}} \int_0^{r_0} \Theta_{\pm}(y,r)\,dr + O_c(\ln(h^{-1})).$$

Paying heed to the equality $\int_0^{\alpha_0} \frac{dx}{1+x^2} = \int_0^{\varphi_0} d\varphi$, we finally obtain

$$\Phi_v(h) = \frac{6}{\pi^2}(K_+(u_+) + K_-(u_-)) \int_0^{\varphi_0} d\varphi + O_{c,\varepsilon}(h^{\frac{1}{2}-\varepsilon}),$$

where

$$K_{\pm}(u_{\pm}) = \int_0^1 \int_0^{r_0} \Theta_{\pm}(y,r)\,dy\,dr.$$

We write $\tau = \frac{1}{r} - (1+v)$. Since

$$\chi_{[r,\infty)}\biggl(\frac{1}{1+v(1-2y)}\biggr) = \chi_{[-2v,\infty)}\biggl(\frac{\tau}{y}\biggr),$$

it follows that

$$K_+(u_+) = \int_0^{r_0} k_+(u_+,r)\,dr,$$

where

$$k_+(u_+,r) = \int_0^1 \chi_{[-2v,\infty)}\biggl(\frac{\tau}{y}\biggr)\biggl(2v + \min\biggl\{u_+ - v,\ \frac{\tau}{y}\biggr\}\biggr)\,dy.$$

By assumption, $u_+ - v \geqslant -2v$. Therefore,

$$\frac{\partial}{\partial u_+} k_+(u_+,r) = \int_0^1 \chi_{[u_+-v,\infty)}\biggl(\frac{\tau}{y}\biggr)\,dy.$$

In the case when $u_+ - v > 0$, this implies that

$$\frac{\partial}{\partial u_+} k_+(u_+,r) = \begin{cases} 1, & \text{if } \tau \geqslant u_+ - v, \\ \dfrac{\tau}{u_+ - v}, & \text{if } 0 \leqslant \tau < u_+ - v, \\ 0, & \text{if } \tau < 0. \end{cases}$$

In the case when $u_+ - v < 0$, we have

$$\frac{\partial}{\partial u_+} k_+(u_+,r) = \begin{cases} 1, & \text{if } \tau \geqslant 0, \\ 1 - \dfrac{\tau}{u_+ - v}, & \text{if } u_+ - v \leqslant \tau < 0, \\ 0, & \text{if } \tau < u_+ - v. \end{cases}$$

Since $k_+(-v,r)=0$, it follows that

$$\frac{6}{\pi^2}\,K_+(u_+)=\int_0^{r_0}\int_{-v}^{u_+}\rho(r,u,v)\,dr\,du.$$

We recall that the function ρ is defined in the theorem in the introduction.

The equality

$$\frac{6}{\pi^2}K_-(u_-)=\int_0^{r_0}\int_{u_-}^{-v}\rho(r,u,v)\,dr\,du$$

can be proved in just the same way. Thus,

$$\Phi_v(h)=\int_0^{\varphi_0}\int_0^{r_0}\int_{u_-}^{u_+}\rho(r,u,v)\,d\varphi\,dr\,du+O_{c,\varepsilon}(h^{\frac{1}{2}-\varepsilon}).$$

This completes the proof of the theorem.

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Received 4/OCT/07
21/JAN/08
Translated by A. I. SHTERN

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