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Minimal Vector Systems in Three-Dimensional Lattices and an Analogue of Vahlen's Theorem for Three-Dimensional Minkowski Continued Fractions

A. V. Ustinov

Khabarovsk Branch, Institute of Applied Mathematics, Far East Division, Russian Academy of Sciences, ul. Zaparina 92, Khabarovsk, 680000 Russia

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To the memory of A. A. Karatsuba

1. INTRODUCTION

There exist two geometric interpretations of classical continued fractions admitting a natural generalization to the multidimensional case. In one of these interpretations, which is due to Klein (see [1, 2]), a continued fraction is identified with the convex hull (the Klein polygon) of the set of integer lattice points belonging to two adjacent angles. The second interpretation, which was independently proposed by Voronoi and Minkowski (see [3–6]), is based on local minima of lattices, minimal systems, and extremal parallelepipeds (the definitions of the notions mentioned in the introduction are given later on). The vertices of Klein polygons in plane lattices can be identified with local minima; however, beginning with the dimension 3, the Klein and Voronoi–Minkowski geometric constructions become different (see [7, 8]).

The constructions of Voronoi and Minkowski is simpler from the computational point of view. In particular, they make it possible to design efficient algorithms for determining fundamental units in cubic fields. In both Voronoi's and Minkowski's approaches, the three-dimensional theory of continued fractions is based on beautiful theorems of the geometry of numbers (a discussion and a reexposition of the original results are contained in monographs [9–11]). Moreover, both of them use the natural assumption that the lattices under consideration are irreducible (that is, none of the coordinates of any two different lattice points coincide). In particular, lattices of cubic irrationalities are irreducible. However, some number-theoretic problems, such as those related to analyzing properties of Korobov grids (see [12–14]), require studying local properties of integer lattices, which are not irreducible.

In [15], it was mentioned that Vahlen's theorem on the approximation of numbers by convergents (see [16, 17]) in terms of lattices admits the following interpretation: If vectors $\gamma_a = (a_1, a_2)$ and $\gamma_b = (-b_1, b_2)$ form a Voronoi basis, then

$$\min\{a_1a_2, b_1b_2\} \le \frac{1}{2}\det\Gamma.$$

In the same paper, the following refinement of Vahlen's theorem was proposed:

$$a_1a_2 + b_1b_2 \le \det\Gamma. \tag{1.1}$$

In [15], Avdeeva and Bykovskii obtained an analogue of inequality (1.1) for Minkowski bases (which are the three-dimensional counterparts of Voronoi bases); namely, they proved that if nodes

$$\gamma_a = (\pm a_1, \pm a_2, \pm a_3), \qquad \gamma_b = (\pm b_1, \pm b_2, \pm b_3), \qquad \text{and} \qquad \gamma_c = (\pm c_1, \pm c_2, \pm c_3), \tag{1.2}$$

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where $a_i, b_i, c_i \geq 0$ for $i = 1, 2, 3$, form a Minkowski basis for a lattice Γ, then

$$a_1a_2a_3 + b_1b_2b_3 + c_1c_2c_3 \leq \det \Gamma. \tag{1.3}$$

(Earlier results related to the three-dimensional version of Vahlen's theorem, as well as some of its refinements, can be found in [15, 18–21].) Inequality (1.3) can be regarded as a sharpening of the estimate

$$a_1a_2a_3 + b_1b_2b_3 + c_1c_2c_3 \leq 3 \det \Gamma, \tag{1.4}$$

which follows from Minkowski's convex body theorem.

In this paper, we suggest an approach to studying minimal systems of vectors in arbitrary lattices. In reducible lattices, the notion of minimality can be given various meanings by imposing more or less strong conditions on the vector system under consideration. Different constraints lead to different three-dimensional analogues of Vahlen's theorem. In the framework of the approach which we suggest in this paper, for minimal systems of the form (1.2), a relaxed version

$$a_1a_2a_3 + b_1b_2b_3 + c_1c_2c_3 \leq 2 \det \Gamma$$

of Vahlen's theorem is valid, while for completely minimal systems (satisfying more rigid conditions), the sharper estimate (1.3) remains valid. This result is based on a complete classification of minimal systems of vectors in arbitrary three-dimensional lattices.

2. MINIMAL SYSTEMS AND MINKOWSKI BASES

2.1. Basic Notions. Let $\gamma_1, \dots, \gamma_t$ be any set of linearly independent vectors in space $\mathbb{R}^s$, $1 \leq t \leq s$. By definition, a *lattice of dimension s and rank t* is a set

$$\Gamma = \langle \gamma_1, \dots, \gamma_t \rangle := \{m_1\gamma_1 + \dots + m_t\gamma_t : m_1, \dots, m_t \in \mathbb{Z}\}.$$

If $t = s$, then such a lattice is said to be *complete*. For a complete lattice Γ, the quantity

$$\det \Gamma = |\det(\gamma_1, \dots, \gamma_s)|$$

does not depend on the choice of the basis $(\gamma_1, \dots, \gamma_s)$ and is called the *lattice determinant*. Given points (nodes) $v_1, \dots, v_s$ in the lattice Γ, the integer

$$I = I(v_1, \dots, v_s) = \frac{|\det(v_1, \dots, v_s)|}{|\det(\gamma_1, \dots, \gamma_s)|} = \frac{|\det(v_1, \dots, v_s)|}{\det \Gamma}$$

is called the *index of the set* $(v_1, \dots, v_s)$ in the lattice Γ. In particular, $I = 1$ if and only if $(v_1, \dots, v_s)$ is a basis in the lattice Γ, and $I = 0$ if and only if the system of vectors $(v_1, \dots, v_s)$ is *degenerate* (linearly dependent).

A lattice Γ is said to be *irreducible* (or *generic*) if the coordinate hyperplanes contain no points of this lattice except the origin; otherwise, the lattice is said to be *reducible*. We denote the set of complete s-dimensional lattices by $\mathcal{L}_s(\mathbb{R})$ and its subset consisting of irreducible lattices by $\mathcal{L}_s^*(\mathbb{R})$.

For a nonempty point set $T \subset \mathbb{R}^s$, we put

$$|T|_i = \max\{|x_i| : x = (x_1, \dots, x_s) \in T\}, \qquad i = 1, \dots, s,$$

$$\Pi(T) = (-|T|_1, |T|_1) \times \dots \times (-|T|_s, |T|_s),$$

and

$$\overline{\Pi}(T) = [-|T|_1, |T|_1] \times \dots \times [-|T|_s, |T|_s].$$

By a *system of rth-order nodes* in a lattice Γ (not necessarily complete) we mean any finite set $(\gamma_1, \dots, \gamma_r)$ of nonzero lattice points in which $\gamma_i \neq \pm\gamma_j$ if $1 \leq i < j \leq r$. To any system $S = (\gamma_1, \dots, \gamma_r)$ we assign the matrix $M(\gamma_1, \dots, \gamma_r)$ whose columns consist of the coordinates of the vectors $\gamma_1, \dots, \gamma_r$.

Let G_s be the group acting on the set of matrices of the form $M(\gamma_1, \dots, \gamma_r)$ and generated by the following elementary transformations:

(1) permutation of rows or columns (renumbering of the vectors in the system under consideration or of the coordinates axes);

(2) changing the signs of all elements in a column (changing the direction of a coordinate axis);

(3) multiplication of a row by a nonzero number (rescaling one of the coordinate axes, possibly in combination with changing the orientation of this axis).

Note that when the rows of the matrix corresponding to a system of vectors in some lattice is multiplied by a number different from ± 1, the same transformation must be applied to the matrix of the lattice basis.

2.2. Minimal Systems in Irreducible Lattices. Local properties of reducible and irreducible lattices may substantially differ. First, we consider generic lattices, because their properties are much simpler.

A point γ in a lattice $\Gamma \in \mathcal{L}_s(\mathbb{R})$ is called a *relative (local) minimum* of the lattice Γ in the sense of Voronoi (or simply a *minimum*) if the closed parallelepiped $\overline{\Pi}(\gamma)$ contains no points of the lattice Γ different from its vertices and the origin (see [4]).

Let Γ be an irreducible, not necessarily complete, lattice of any dimension. A system S in the lattice Γ is said to be *minimal* if the parallelepiped $\Pi(S)$ contains no points of Γ except the origin. In particular, for irreducible lattices, the notion of a minimal system of order 1 coincides with that of a local minimum. For reducible lattices, the definition of a minimal system must be refined.

For a minimal system S, the parallelepiped $\Pi(S)$ is said to be *extremal*: it is impossible to extend this parallelepiped in any coordinate direction so that the resulting parallelepiped still contains no nonzero lattice points.

If $\gamma \in \Pi(\gamma_1, \ldots, \gamma_t)$, then we say that the point γ *violates the minimality of the system* $(\gamma_1, \ldots, \gamma_t)$.

In irreducible lattices, each face of the parallelepiped $\Pi(\gamma_1, \ldots, \gamma_t)$ contains precisely one point from the minimal system $S = (\gamma_1, \ldots, \gamma_t)$; therefore, the rank t of the system S never exceeds the dimension s.

In the two-dimensional case, Voronoi proved by translating the classical theory of continued fractions into geometric language (see [4]) that any minimal system $\{\gamma_a, \gamma_b\}$ of two vectors in a two-dimensional lattice Γ is a basis, and the action of the group G_2 reduces the matrix $M(\gamma_a, \gamma_b)$ to the form

$$\begin{pmatrix} 1 & -b_1 \\ a_2 & 1 \end{pmatrix}, \qquad a_2, b_1 \in [0, 1]. \tag{2.1}$$

Obviously, the converse is also true: a pair of vectors (γ_a, γ_b) for which the matrix $M(\gamma_a, \gamma_b)$ is equivalent to (2.1) forms a minimal system in the lattice $\Gamma = \langle \gamma_a, \gamma_b \rangle$. We refer to those bases of two-dimensional lattices which are minimal systems as *Voronoi bases*.

In the three-dimensional case, given a minimal system consisting of three vectors, we denote these vectors by γ_a, γ_b, and γ_c and write them in the form (1.2). To a minimal system $S = (\gamma_a, \gamma_b, \gamma_c)$ we assign the matrix

$$M(S) = M(\gamma_a, \gamma_b, \gamma_c) = \begin{pmatrix} \pm a_1 & \pm b_1 & \pm c_1 \\ \pm a_2 & \pm b_2 & \pm c_2 \\ \pm a_3 & \pm b_3 & \pm c_3 \end{pmatrix}, \tag{2.2}$$

whose columns are formed by the coordinates of the vectors in the system.

A minimal system $(\gamma_a, \gamma_b, \gamma_c)$ is said to be *degenerate* if $\det M(\gamma_a, \gamma_b, \gamma_c) = 0$; otherwise, this system is said to be *nondegenerate*.

The group G_3 preserves the local structure of the lattice. Matrices which are mapped to each other under the action of the group G_3 are said to be *equivalent*.

The set of matrices of minimal systems consisting of three vectors decomposes into three orbits under the action of the group G_3. Each face of the extremal parallelepiped $\Pi(\gamma_a, \gamma_b, \gamma_c)$ contains precisely one of the points $\pm\gamma_a$, $\pm\gamma_b$, and $\pm\gamma_c$. Therefore, the coordinate axes can be defined so that, in each row of the matrix $M(\gamma_a,\gamma_b,\gamma_c)$, the element with maximum absolute value is positive and belongs to the main diagonal. Rescaling the axes, we can make the diagonal elements of the matrix $M(\gamma_a,\gamma_b,\gamma_c)$ to equal 1. Thus, the problem reduces to analyzing minimal systems with matrix of the *standard form*

$$\begin{pmatrix} 1 & \pm b_1 & \pm c_1 \\ \pm a_2 & 1 & \pm c_2 \\ \pm a_3 & \pm b_3 & 1 \end{pmatrix}, \qquad a_2, a_3, b_1, b_3, c_1, c_2 \in [0,1]. \tag{2.3}$$

The above normalization transforms the extremal parallelepiped $\Pi(\gamma_a,\gamma_b,\gamma_c)$ into the unit cube $\Pi := (-1,1)^3$. Note that, for irreducible lattices, the elements of matrix (2.3) satisfy the conditions $a_2, a_3, b_1, b_3, c_1, c_2 \in (0,1)$.

Two vectors belonging to the same minimal system cannot have equal (or opposite) signs, because if they have, then their difference (or sum) violates the minimality of the system. This leads to 24 possible arrangements of signs in matrix (2.3) (see Table 1).

Table 1

1 (1, II)	2 (2, II)	3 (3, II)	4 (4, I)	5 (5, I)	6 (6, III)
+ − +	+ + +	+ + −	+ + −	+ − +	+ − −
+ + +	− + +	+ + +	− + +	+ + −	− + −
− + +	+ − +	+ − +	+ − +	− + +	− − +

7 (1, II)	8 (2, II)	9 (3, II)	10 (4, I)	11 (5, I)	12 (6, III)
+ + −	+ − +	+ + +	+ − −	+ + +	+ + −
− + +	+ + −	+ + −	+ + −	− + +	+ + +
+ + +	+ + +	− + +	+ + +	− − +	− + +

13 (1, II)	14 (2, II)	15 (3, II)	16 (4, I)	17 (5, I)	18 (6, III)
+ + +	+ + −	+ − +	+ + +	+ − −	+ + +
− + −	− + −	− + +	− + −	+ + +	+ + −
− − +	− + +	− − +	− + +	+ − +	+ − +

19 (1, II)	20 (2, II)	21 (3, II)	22 (4, I)	23 (5, I)	24 (6, III)
+ − −	+ − −	+ − −	+ − +	+ + −	+ − +
+ + −	+ + +	− + −	+ + +	− + −	− + +
+ − +	− − +	+ + +	− − +	+ + +	+ + +

The Arabic numeral in parentheses following the number of an arrangement of signs indicates the number of the equivalent signature from Table 2, and the Roman numeral is the type of the corresponding minimal triple (the definition is given below).

Changing signs in rows and columns, we can reduce these 24 cases to the six signatures given in Table 2.

Table 2. Normal signatures and the corresponding types of minimal triples

1 (II)	2 (II)	3 (II)	4 (I)	5 (I)	6 (III)
$+-+$	$+++$	$++-$	$++-$	$+-+$	$+--$
$+++$	$-++$	$+++$	$-++$	$++-$	$-+-$
$-++$	$+-+$	$+-+$	$+-+$	$-++$	$--+$

Permuting rows and columns so as to preserve the diagonal dominance of matrices, we further reduce the number of possible configurations to three. Thus, the set of all matrices of minimal systems decomposes into three classes of matrices equivalent with respect to the action of the group G_3, which have representatives of the forms

$$\begin{pmatrix} 1 & b_1 & -c_1 \\ -a_2 & 1 & c_2 \\ a_3 & -b_3 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & b_1 & c_1 \\ -a_2 & 1 & c_2 \\ a_3 & -b_3 & 1 \end{pmatrix}, \qquad \text{and} \qquad \begin{pmatrix} 1 & -b_1 & -c_1 \\ -a_2 & 1 & -c_2 \\ -a_3 & -b_3 & 1 \end{pmatrix},$$

where $a_i, b_i, c_i \in [0, 1]$.

These three matrices correspond to minimal triples of three types, which can be regarded as three-dimensional analogues of the Voronoi bases. A precise description of the minimal triples is given by the following result of Minkowski.

Theorem 4 (Minkowski). *Let $S = (\gamma_a, \gamma_b, \gamma_c)$ be a minimal system in a lattice Γ. If the system S is nondegenerate, then the triple of vectors $(\gamma_a, \gamma_b, \gamma_c)$ is a basis in the lattice Γ and the matrix $M(\gamma_a, \gamma_b, \gamma_c)$ is equivalent to one of the two canonical forms*

$$M_1 = \begin{pmatrix} 1 & b_1 & -c_1 \\ -a_2 & 1 & c_2 \\ a_3 & -b_3 & 1 \end{pmatrix}, \qquad \begin{gathered} a_2, a_3, b_2, b_3, c_1, c_2 \in [0, 1], \\ c_1 \le b_1, \end{gathered} \tag{2.4}$$

and

$$M_2 = \begin{pmatrix} 1 & b_1 & c_1 \\ -a_2 & 1 & c_2 \\ a_3 & -b_3 & 1 \end{pmatrix}, \qquad \begin{gathered} a_2, a_3, b_2, b_3, c_1, c_2 \in [0, 1], \\ c_1 \le b_1, \quad a_2 + c_2 \ge 1. \end{gathered} \tag{2.5}$$

If the system S is degenerate, then $\gamma_a \pm \gamma_b \pm \gamma_c = 0$ for some combination of signs, and the matrix $M(\gamma_a, \gamma_b, \gamma_c)$ is reduced by the action of the group G_3 to the form

$$M_3 = \begin{pmatrix} 1 & -b_1 & -c_1 \\ -a_2 & 1 & -c_2 \\ -a_3 & -b_3 & 1 \end{pmatrix}, \qquad \begin{gathered} a_2, a_3, b_2, b_3, c_1, c_2 \in [0, 1], \\ b_1 + c_1 = 1, \quad a_2 + c_2 = 1, \quad a_3 + b_3 = 1. \end{gathered} \tag{2.6}$$

The converse is also true: a system of three vectors $(\gamma_a, \gamma_b, \gamma_c)$ whose matrix is equivalent to a matrix of one of the forms (2.4) *and* (2.5) *is a minimal system in the complete lattice $\Gamma = \langle \gamma_a, \gamma_b, \gamma_c \rangle$, and the system of vectors $(\gamma_a, \gamma_b, \gamma_c)$ with matrix of the form* (2.6) *is a minimal system for the (rank-2) lattice $\Gamma = \langle \gamma_a, \gamma_b, \gamma_c \rangle = \langle \gamma_a, \gamma_b \rangle$.*

Minkowski stated this theorem without proof (see [5, 6]). A detailed proof can be found in [11, Art. 109–110] (see also [18, 22–24]).

Having in mind Theorem 4, we refer to minimal systems whose matrices are equivalent to matrices (2.4) and (2.5) as *Minkowski bases of types* I *and* II, respectively, and to minimal systems of the form (2.6) as *degenerate Minkowski triples*, or *triples of type* III.

Remark 1. Minkowski wrote the inequalities for the elements of matrices (2.4)–(2.5) in a more symmetric form. Namely, for the matrix M_1, he wrote

$$c_1 \le b_1 \quad \text{or} \quad a_2 \le c_2 \quad \text{or} \quad b_3 \le a_3,$$

and for the matrix M_2, he wrote $a_2 + c_2 \ge 1$ and

$$c_1 \le b_1 \quad \text{or} \quad a_3 \le b_3.$$

The simpler, although less symmetric, conditions in (2.4) and (2.5) are obtained from these inequalities by applying equivalent transformations; namely, for matrix (2.4), it suffices to cyclically permute all rows and columns, and for matrix (2.5), to interchange the first and third columns and the first and third rows (the signature number 2 in Table 1 is equivalent to the signature number 8).

3. MINIMAL SYSTEMS IN REDUCIBLE LATTICES

The definition of minimal systems given above for irreducible lattices can be directly transferred to any lattices. However, this approach involves some difficulties, which leads to an unnecessary complication of the situation. In some lattices, there exist minimal systems consisting of more than three vectors. The convex hulls of such systems may contain lattice points on edges and inside faces. The index of a minimal system may attain the values 3 and 4. (For example, this happens for the lattice $\mathbb{Z}^3$: the surface of the cube $[-1, 1]^3$ contains a minimal system of 13 vectors, some of which are midpoints of edges or central points of faces.) Moreover, the classification of minimal systems becomes very cumbersome. The three-dimensional analogue of Vahlen's theorem is no longer valid; the example of the triple of vectors $\gamma_a = (-1, 1, 1)$, $\gamma_b = (1, -1, 1)$, and $\gamma_c = (1, 1, -1)$ in the lattice $\mathbb{Z}^3$ shows that nothing better than the trivial estimate (1.4) can be proved.

The situation becomes more natural when the definition of minimal systems is supplemented by an additional condition.

Suppose given a triple of linearly independent vectors $\gamma_a, \gamma_b, \gamma_c \in \Gamma$. We say that the octahedron with vertices $\pm\gamma_a$, $\pm\gamma_b$, $\pm\gamma_c$ is *empty* if its (strict) interior contains no points of the lattice Γ except the origin. An empty octahedron whose surface contains points of the lattice Γ different from $\pm\gamma_a$, $\pm\gamma_b$, and $\pm\gamma_c$ is said to be *primitive*.

For any lattice $\Gamma \in \mathcal{L}_3(\mathbb{R})$, we say that a system of vectors $(\gamma_1, \dots, \gamma_t)$ is *minimal* if the following conditions hold:

(1) the system consists of at most three vectors;

(2) the parallelepiped $\Pi(\gamma_1, \dots, \gamma_t)$ contains no points of the lattice Γ except the origin;

(3) if $t = 3$ and the system $(\gamma_1, \dots, \gamma_t)$ is nondegenerate, then the octahedron with vertices $\pm\gamma_1$, $\pm\gamma_2$, and $\pm\gamma_3$ is primitive.

For generic lattices, the first and last conditions hold automatically; thus, the new definition does not contradict the initial one.

Studying the more general situation where the triple of vectors under consideration satisfies the first two requirements but does not necessarily satisfy the last one reduces to analyzing minimal systems by using Theorem 6 (see below).

In analyzing minimal systems, we assume that each row contains a nonzero element, because otherwise the problem becomes planar, and a complete description reduces to Voronoi bases (2.1). As in the case of irreducible lattices, representatives of equivalence classes are usually chosen so that, in all rows, the elements with maximal absolute value are equal to 1. By the definition of a minimal system, each column of the matrix $M(\gamma_a, \gamma_b, \gamma_c)$ also must contain an element with absolute value 1. Thus, the vectors forming a minimal system always belong to the surface of the cube $\Pi = (-1, 1)^3$.

Under the passage from irreducible lattices to reducible ones, the structure of the set of minimal systems becomes substantially more complicated. To emphasize the difference, we compare properties of minimal systems in irreducible and reducible lattices.

Properties of Minimal Systems in Irreducible Lattices

1°. Two vectors from the same minimal system cannot belong to the same octant (octants are assumed to be closed; opposite octants are identified). In other words, two vectors from a minimal system cannot have equal or opposite signatures.

2°. If $(\gamma_a, \gamma_b, \gamma_c)$ is a minimal system in a lattice Γ, then each face of the extremal parallelepiped $\Pi(\gamma_a, \gamma_b, \gamma_c)$ contains precisely one lattice point. The matrix $M(\gamma_a, \gamma_b, \gamma_c)$ can be reduced to the standard form (2.3) (that is, to a matrix with diagonal dominance) by elementary transformations, and the signatures of the vectors $\pm\gamma_a$, $\pm\gamma_b$, and $\pm\gamma_c$ are pairwise different.

3°. Minimal triples satisfy the conditions of Minkowski's theorem 4; i.e., the matrix $M(\gamma_a, \gamma_b, \gamma_c)$ is equivalent to one of matrices (2.4)–(2.6) with elements satisfying the corresponding constraints. In particular, the index of a minimal triple can equal 0 or 1.

4°. Each vector of a minimal system is a local minimum of the lattice.

Properties of Minimal Systems in Reducible Lattices

1°. Two vectors from the same minimal system may belong to the same octant. Moreover, there exist minimal systems of three elements with the following property: under any assignment of signs to zero coordinates, the vectors of the system are contained in the union of two octants. In what follows, we refer to such systems as systems *lying in two octants*. (See examples in Theorem 9.)

2°. Two vectors from a minimal system $(\gamma_a, \gamma_b, \gamma_c)$ may be contained strictly inside the same face of the extremal parallelepiped $\Pi(\gamma_a, \gamma_b, \gamma_c)$. Accordingly, the matrix $M(\gamma_a, \gamma_b, \gamma_c)$ is not necessarily equivalent to a matrix of the form (2.3). We refer to such systems as *systems without diagonal dominance*. (See examples in Theorem 10.)

3°. Even in the case where the vectors of a minimal system $S = (\gamma_a, \gamma_b, \gamma_c)$ belong to three pairwise different octants and the action of the group G_3 reduces the matrix $M(\gamma_a, \gamma_b, \gamma_c)$ to the standard form (2.3), it may happen that Minkowski's theorem 4 in not valid for S: triples with the same signs as in matrices (2.4) and (2.5) are not necessarily bases and may have index 0 or 2; a triple $(\gamma_a, \gamma_b, \gamma_c)$ with the same signs as in matrix (2.6) may be a basis of the lattice, or it may be a degenerate system for which $\gamma_a + \gamma_b + \gamma_c \neq 0$. (See examples in Theorems 11–13.)

4°. A vector of a minimal system is not necessarily a local minimum of the lattice.

We refer to a minimal system in which every vector is a minimum of the lattice as *completely minimal*. For irreducible lattices, the notions of a minimal system and a completely minimal system are equivalent. For reducible lattices, the class of completely minimal systems is narrower. For example, in the lattice $\mathbb{Z}^3$, the system of three vectors $(0, 0, 1)$, $(0, 1, 1)$, and $(1, 1, 1)$ is minimal but not completely minimal.

We shall prove (see Theorem 14) that, for completely minimal systems in any lattices, inequality (1.3) from the three-dimensional analogue of Vahlen's theorem remains valid. For the set of all minimal systems, a weakened version of inequality (1.3) holds; this version is, however, sharper than the trivial estimate (1.4).

We refer to minimal triples whose matrices can be reduced to the forms (2.4)–(2.6) as *minimal Minkowski systems* (these are *Minkowski bases of types* I *and* II and *degenerate minimal Minkowski systems*, respectively); the remaining minimal triples are said to be *exceptional*. According to the above considerations, the exceptional minimal triples can be divided into the following three categories (with nonempty intersections):

- triples lying in two octants;
- triples without diagonal dominance;
- exceptional triples of the standard form (2.3).

Inside each category, triples are distinguished by their indices, which can take values from 0 to 2 by Theorem 5 (see below).

In what follows, for each type of triples, we obtain a description similar to the description of minimal Minkowski systems given by Theorem 4. In all, we obtain 12 types of minimal triples. A description of nine types of exceptional triples is given by Theorems 9–13 below.

4. MAIN TOOLS: MINKOWSKI'S THEOREMS AND THEIR GENERALIZATIONS

To analyze minimal systems, we need several auxiliary assertions about properties of three-dimensional lattices.

Theorem 5 (Minkowski). *Given a primitive octahedron with vertices $\pm\gamma_a, \pm\gamma_b, \pm\gamma_c \in \Gamma$, either the set $(\gamma_a, \gamma_b, \gamma_c)$ is a basis of the lattice Γ or $(\gamma_a, \gamma_b, \gamma_c)$ is a system of index 2 in the lattice Γ and $\Gamma = \langle \gamma_a, \gamma_b, (\gamma_a + \gamma_b + \gamma_c)/2 \rangle$.*

The proof of this theorem can be found in [25, pp. 97–101] and [11, Art. 161].

As mentioned above, a similar assertion for empty but not primitive octahedra makes it possible to describe configurations of lattice points on the surface of any extremal parallelepiped.

Theorem 6. *Suppose that an octahedron with vertices $\pm\gamma_a, \pm\gamma_b, \pm\gamma_c \in \Gamma$ is empty but not primitive. Then one of the following three possibilities is realized (see Fig. 1):*

(1) *$I(\gamma_a, \gamma_b, \gamma_c) = 2$ and one of the sets*

$$\left(\gamma_a, \gamma_b, \frac{\gamma_a \pm \gamma_b}{2}\right), \qquad \left(\gamma_b, \gamma_c, \frac{\gamma_b \pm \gamma_c}{2}\right), \qquad \textit{and} \qquad \left(\gamma_c, \gamma_a, \frac{\gamma_c \pm \gamma_a}{2}\right)$$

is a basis of the lattice Γ;

(2) *$I(\gamma_a, \gamma_b, \gamma_c) = 3$ and one of the sets*

$$\left(\gamma_a, \gamma_b, \frac{\gamma_a \pm \gamma_b \pm \gamma_c}{3}\right)$$

is a basis of the lattice Γ;

(3) *$I(\gamma_a, \gamma_b, \gamma_c) = 4$ and one of the sets*

$$\left(\gamma_a, \gamma_b, \frac{2\gamma_a \pm \gamma_b \pm \gamma_c}{4}\right), \quad \left(\gamma_b, \gamma_c, \frac{2\gamma_b \pm \gamma_c \pm \gamma_a}{4}\right),$$
$$\left(\gamma_c, \gamma_a, \frac{2\gamma_c \pm \gamma_a \pm \gamma_b}{4}\right), \quad \textit{and} \quad \left(\frac{\gamma_a + \gamma_b}{2}, \frac{\gamma_b + \gamma_c}{2}, \frac{\gamma_a + \gamma_a}{2}\right)$$

is a basis of the lattice Γ.

The proof of Theorem 6 uses the following assertion (see [26, Lemma 2]).

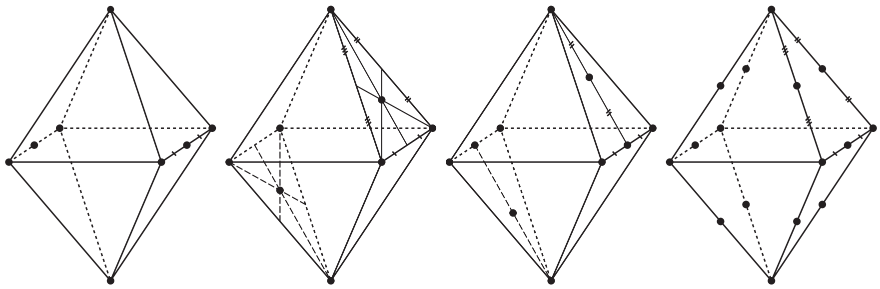

Fig. 1. Possible configurations of lattice points on the surface of an empty but not primitive octahedron

Lemma 6. *If α_1, α_2, and α_3 are real numbers and $\|\alpha_1\| \le \|\alpha_2\| \le \|\alpha_3\|$, then*

$$\min\{\|\alpha_1\| + \|\alpha_2\| + \|\alpha_3\|, \|2\alpha_1\| + \|2\alpha_2\| + \|2\alpha_3\|\} \le 1.$$

Moreover, if

$$\min\{\|\alpha_1\| + \|\alpha_2\| + \|\alpha_3\|, \|2\alpha_1\| + \|2\alpha_2\| + \|2\alpha_3\|\} = 1,$$

then

$$\|3\alpha_1\| + \|3\alpha_2\| + \|3\alpha_3\| \le 1,$$

and the equality can occur only in the case where $\|\alpha_1\| = 1/4$, $\|\alpha_2\| = 1/4$, and $\|\alpha_3\| = 1/2$.

Proof (of Theorem 6)**.** Since the given octahedron is not primitive, it follows that $I(\gamma_a, \gamma_b, \gamma_c) \ge 2$. Hence, there exists a nonzero lattice point

$$\gamma = k_a\gamma_a + k_b\gamma_b + k_c\gamma_c$$

for which $k_a, k_b, k_c \in (-1/2, 1/2]$. We assume that the vectors γ_a, γ_b, and γ_c are ordered so that $\|k_a\| \le \|k_b\| \le \|k_c\|$. Reducing the coefficients k_a, k_b, and k_c to a common denominator, we can write them in the form

$$k_a = \frac{m_a}{n}, \quad k_b = \frac{m_b}{n}, \quad \text{and} \quad k_c = \frac{m_c}{n}, \qquad n > 1.$$

If there are several such vectors γ, then we choose the one with largest n.

The vector

$$\gamma_l = \|lk_a\|\gamma_a + \|lk_b\|\gamma_b + \|lk_c\|\gamma_c, \qquad l = 1, 2, \dots,$$

does not belong to the interior of the octahedron with vertices $\pm\gamma_a$, $\pm\gamma_b$, and $\pm\gamma_c$ if and only if

$$\|lk_a\| + \|lk_b\| + \|lk_c\| \ge 1.$$

By Lemma 6, the vector $(\|k_a\|, \|k_b\|, \|k_c\|)$ must coincide with one of the sets

$$\left(0, \frac{1}{2}, \frac{1}{2}\right), \quad \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \quad \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \quad \text{and} \quad \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right). \tag{4.1}$$

Under the assumption that there exists a lattice point

$$\gamma' = k'_a\gamma_a + k'_b\gamma_b + k'_c\gamma_c$$

such that $k'_a, k'_b, k'_c \in (-1/2, 1/2]$ and

$$(k'_a, k'_b, k'_c) \not\equiv (lk_a, lk_b, lk_c) \pmod 1, \qquad l = 1, 2, \dots,$$

the set $(\|k'_a\|, \|k'_b\|, \|k'_c\|)$ must coincide with one of the sets in (4.1) up to permutation. If the set $(\|k'_a\|, \|k'_b\|, \|k'_c\|)$ is not equal to one of the sets $(1/2, 0, 1/2)$ and $(1/2, 1/2, 0)$, then, considering linear combinations of the vectors γ, γ', γ_a, γ_b, and γ_c, we can obtain a vector with coefficients such that neither these coefficients nor their permutations are contained in list (4.1). This contradiction shows that either $\Gamma = \langle \gamma, \gamma_b, \gamma_c \rangle$ or

$$\Gamma = \left\langle \frac{\gamma_a + \gamma_b}{2}, \frac{\gamma_a + \gamma_c}{2}, \frac{\gamma_b + \gamma_c}{2} \right\rangle.$$

The set of coefficients $(1/2, 1/2, 1/2)$ is redundant, because the corresponding octahedron is primitive. The remaining possibilities lead to the configurations described in the statement of the theorem.

Theorem 7 (Minkowski). *Let Ω be a convex body centrally symmetric about the origin. Suppose that $\pm\gamma_a, \pm\gamma_b, \pm\gamma_c \in \Gamma$ are the vertices of a primitive octahedron lying on the surface Ω.*

(1) *If $|\det(\gamma_a, \gamma_b, \gamma_c)| = \det \Gamma$, then the interior of the domain Ω does not contain points of the lattice Γ different from the origin if and only if the interior of Ω contains no linear combinations of the points γ_a, γ_b, and γ_c with coefficients from the following 22 sets:*

$$\begin{gathered} (1, \pm 1, 0), \quad (1, 0, \pm 1), \quad (0, 1, \pm 1), \quad (1, \pm 1, \pm 1), \\ (2, \pm 1, \pm 1), \quad (\pm 1, 2, \pm 1), \quad (\pm 1, \pm 1, 2). \end{gathered} \tag{4.2}$$

(2) *If $|\det(\gamma_a, \gamma_b, \gamma_c)| = 2 \det \Gamma$, then the interior of the domain Ω contains no points of the lattice Γ different from the origin if and only if the interior Ω contains no linear combinations of the points γ_a, γ_b, and γ_c with coefficients*

$$\left(\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2} \right). \tag{4.3}$$

The proof of this theorem is given in [25, pp. 101–104], [27, pp. 12–14], and [11, Art. 162].

Theorem 7 can be supplemented by the following assertion, which we shall apply to analyze degenerate triples.

Theorem 8. *Let Ω be a convex body centrally symmetric about the origin. Suppose that the points of a system $(\gamma_a, \gamma_b, \gamma_c)$ in lattice Γ lie on the surface Ω and $\det(\gamma_a, \gamma_b, \gamma_c) = 0$.*

(1) *The interior of the domain Ω contains no points of the lattice Γ different from the origin if and only if the interior of Ω contains no nonzero linear combinations of the points γ_a, γ_b, and γ_c with coefficients from the list*

$$(1, \pm 1, 0), \quad (1, 0, \pm 1), \quad (0, 1, \pm 1), \quad (1, \pm 1, \pm 1). \tag{4.4}$$

(2) *If the interior of the domain Ω does not contain points of the lattice Γ different from the origin, then among the linear combinations of the vectors γ_a, γ_b, and γ_c with coefficients from the list*

$$(1, \pm 1, \pm 1), \quad (2, \pm 1, \pm 1), \quad (\pm 1, 2, \pm 1), \quad (\pm 1, \pm 1, 2) \tag{4.5}$$

there is the zero vector.

Proof. (1) The necessity of the conditions specified in the statement of the theorem is obvious. Let us prove their sufficiency. We draw a plane π through the points γ_a, γ_b, and γ_c and set $\Gamma' = \Gamma \cap \pi$. Let M be the convex hull of $\pm\gamma_a$, $\pm\gamma_b$, and $\pm\gamma_c$. Points of the lattice Γ' may occur in the polygon M only on its boundary. If there are n such points, then the area of M equals $(n/2) \det \Gamma'$. By Minkowski's convex body theorem, this area does not exceed $4 \det \Gamma'$, i.e., $n \leq 8$. Therefore, M is either a hexagon or a parallelogram, and the midpoints of its edges are points of the lattice Γ' as well. We assume that γ_a, γ_b, and γ_c are consecutive lattice points on the perimeter of M (this can always be achieved by changing the orientation of the vectors). For the domain $\Omega' = \Omega \cap \pi$ to contain no points of the lattice Γ, it is sufficient that

$$\gamma_a + \gamma_b, \gamma_b + \gamma_c, \gamma_c - \gamma_a \notin \Omega'$$

(in the case of a hexagon) or

$$\gamma_a + \gamma_b - \gamma_c, \gamma_c - \gamma_a, \gamma_b + \gamma_c - \gamma_a \notin \Omega'$$

(in the case of a parallelogram). Thus, in both cases, it suffices to check the linear combinations of the vectors γ_a, γ_b, and γ_c with coefficients (4.4).

(2) Suppose again that γ_a, γ_b, and γ_c are consecutive points of the lattice on the perimeter of the convex hull M of $\pm\gamma_a$, $\pm\gamma_b$, and $\pm\gamma_c$. If M is a hexagon, then $\gamma_a - \gamma_b + \gamma_c = 0$. If M is a parallelogram, then $\gamma_a - 2\gamma_b + \gamma_c = 0$. In both cases, it suffices to check the linear combinations of the vectors γ_a, γ_b, and γ_c with coefficients (4.5).

5. CLASSIFICATION OF EXCEPTIONAL MINIMAL SYSTEMS

Theorems 9–13 give a description of exceptional minimal triples. Their proofs follow approximately the same scheme, which is borrowed from the proof of Minkowski's theorem 4. For nondegenerate minimal systems S, we shall find necessary and sufficient conditions on the coefficients of the matrix $M(S)$ by using Theorem 7, and for degenerate systems, by using Theorem 8.

Assigning signs to elements of matrices of minimal systems, we assume that the zero elements can be assigned any sign.

5.1. Minimal Systems Lying in Two Octants.

Theorem 9. *Let $S = (\gamma_a, \gamma_b, \gamma_c)$ be a minimal system in a lattice Γ such that the vectors $\pm\gamma_a$, $\pm\gamma_b$, and $\pm\gamma_c$ are contained in the union of two octants under any arrangement of signs. If S is a degenerate triple, then the matrix $M(\gamma_a, \gamma_b, \gamma_c)$ is equivalent to one of the two matrices (see Fig. 2)*

$$M_4 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & b_2 & b_2 - 1 \\ 1 & b_3 & b_3 - 1 \end{pmatrix}, \qquad b_2, b_3 \in (0,1), \tag{5.1}$$

and

$$M_5 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & -1 \\ \frac{1+c_2}{2} & 1 & -c_3 \end{pmatrix}, \qquad c_3 \in (0,1]. \tag{5.2}$$

If S is a nondegenerate triple, then it is a basis for the lattice Γ, and the matrix $M(\gamma_a, \gamma_b, \gamma_c)$ is equivalent to a matrix of the form

$$M_6 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & b_2 & -c_2 \\ a_3 & 1 & -c_3 \end{pmatrix}, \qquad \begin{gathered} a_3, b_2, c_2, c_3 \in (0,1], \\ 2a_3 \le c_3 \ \text{or} \ a_3 \le c_3, \ \ b_2 + c_2 \le 1. \end{gathered} \tag{5.3}$$

The converse is also true: any system of three vectors $(\gamma_a, \gamma_b, \gamma_c)$ whose matrix is equivalent to a matrix of the form (5.1) *or* (5.2) *is a minimal system in the lattice (of rank* 2*) $\Gamma = \langle\gamma_a, \gamma_b, \gamma_c\rangle = \langle\gamma_b, \gamma_c\rangle$; the system of vectors $(\gamma_a, \gamma_b, \gamma_c)$ with matrix of the form* (5.3) *is a minimal system in the complete lattice $\Gamma = \langle\gamma_a, \gamma_b, \gamma_c\rangle$. In any case, there exists no orientation of the vectors γ_a, γ_b, and γ_c under which these vectors belong to three pairwise different octants.*

Proof. If the vectors γ_a, γ_b, and γ_c are contained in two octants, then we can choose a pair of vectors among them which are contained in the same octant. Suppose that these are the vectors γ_a and γ_b and they belong to the octant $\{(x_1, x_2, x_3) : x_1, x_2, x_3 \ge 0\}$. Their difference $\gamma_a - \gamma_b = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$ does not violate the minimality of the given system S if one of the following conditions holds:

$$|a_1 - b_1| = 1, \qquad |a_2 - b_2| = 1, \qquad \text{and} \qquad |a_3 - b_3| = 1.$$

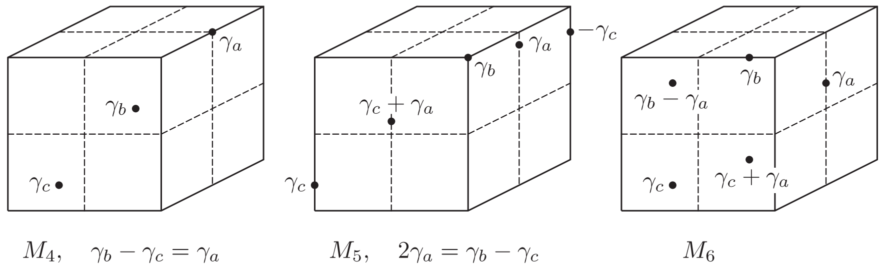

Fig. 2. The arrangement of the points of minimal systems with matrices M_4–M_6

Without loss of generality, we assume that $|a_1 - b_1| = 1$. The equalities $a_1 - b_1 = 1$ and $a_1 - b_1 = -1$ are equivalent (they can be reduced to each other by applying the action of the group G_3); therefore, it suffices to consider only the case where $a_1 - b_1 = -1$, i.e., $a_1 = 0$ and $b_1 = 1$. One of the coordinates of the vector γ_a must equal 1; hence, we can also assume that $a_2 = 1$. Moreover, we have $a_3 > 0$, because at $a_3 = 0$, the vector γ_a belongs to four octants simultaneously, and under some arrangement of signs, the vectors $\pm\gamma_a$, $\pm\gamma_b$, and $\pm\gamma_c$ fall in pairwise different octants. Thus, it is sufficient to consider only matrices of the form

$$\begin{pmatrix} 0 & 1 & \pm c_1 \\ 1 & b_2 & \pm c_2 \\ a_3 & b_3 & \pm c_3 \end{pmatrix}, \qquad a_3 > 0.$$

Since $a_1 = 0$, we can assign the signature $(-++)$ to the vector γ_a and assume that γ_a and γ_b belong to different octants. By the assumptions of the theorem, γ_c belongs to the union of these octants; moreover, it cannot have nonnegative coordinates. Therefore, γ_c must have the form $\gamma_c = (c_1, -c_2, -c_3)$. Recall that each row of the matrix $M(\gamma_a, \gamma_b, \gamma_c)$ must contain an element with absolute value 1. There exist only three matrices satisfying these conditions:

$$\begin{pmatrix} 0 & 1 & c_1 \\ 1 & b_2 & -c_2 \\ 1 & b_3 & -c_3 \end{pmatrix}, \tag{5.4}$$

$$\begin{pmatrix} 0 & 1 & c_1 \\ 1 & b_2 & -c_2 \\ a_3 & 1 & -c_3 \end{pmatrix}, \tag{5.5}$$

and

$$\begin{pmatrix} 0 & 1 & c_1 \\ 1 & b_2 & -c_2 \\ a_3 & b_3 & -1 \end{pmatrix}. \tag{5.6}$$

Note that the elements a_3, b_2, b_3, c_2, and c_3 cannot be zero, because otherwise, changing the sign of the zero element, we obtain a triple of vectors belonging to pairwise different octants. It follows that, for each of the three matrices (5.4)–(5.6), the vector $\gamma_a + \gamma_c = (c_1, \dots)$ does not violate the minimality of the given system S only if $c_1 = 1$.

First, suppose that the system S is degenerate. Considering linear combinations of the vectors γ_a, γ_b, and γ_c with coefficients (4.5), we see that, for a matrix of the form (5.4), only the vector

$$\gamma_a - \gamma_b + \gamma_c = (0, 1 - b_2 - c_2, 1 - b_3 - c_3)$$

can vanish. In this case, $c_2 = 1 - b_2$ and $c_3 = 1 - b_3$, and we obtain triples of the form (5.1). For a matrix of the form (5.5), only the combination

$$2\gamma_a - \gamma_b + \gamma_c = (0, 2 - b_2 - c_2, 2a_3 - 1 - c_3)$$

can vanish. If it does vanish, then $b_2 = c_2 = 1$ and $a_3 = (c_3 + 1)/2$. Therefore, the matrix of the given minimal system is equivalent to (5.2). For a matrix of the form (5.6), only the combination

$$2\gamma_a - \gamma_b - \gamma_c = (0, 2 - b_2 - c_2, 2a_3 - b_3 - 1)$$

can vanish. In this case, $b_2 = c_2 = 1$ and $a_3 = (b_3 + 1)/2$, and we obtain a matrix equivalent to (5.2).

Now, we shall assume that S is a nondegenerate system. To obtain necessary and sufficient conditions on the elements of the matrix $M(\gamma_a, \gamma_b, \gamma_c)$, we apply Theorem 7.

Let $S = (\gamma_a, \gamma_b, \gamma_c)$ be a minimal system with matrix (5.4). Considering linear combinations of the vectors γ_a, γ_b, and γ_c with coefficients (4.2), we see that additional constraints on the elements of the matrix $M(\gamma_a, \gamma_b, \gamma_c)$ arise only because of the vectors

$$\gamma_b - \gamma_c = (0, b_2 + c_2, b_3 + c_3),$$
$$\gamma_a - \gamma_b + \gamma_c = (0, 1 - b_2 - c_2, 1 - b_3 - c_3),$$

and

$$2\gamma_a - \gamma_b + \gamma_c = (0, 2 - b_2 - c_2, 2 - b_3 - c_3).$$

The conditions that these vectors do not violate the minimality of the system S can be written as

$$b_2 + c_2 \geq 1 \quad \text{or} \quad b_3 + c_3 \geq 1,$$
$$b_2 = c_2 = 1 \quad \text{or} \quad b_3 = c_3 = 1,$$

and

$$b_2 + c_2 \leq 1 \quad \text{or} \quad b_3 + c_3 \leq 1,$$

respectively. The symmetry of the situation allows us to assume that $b_3 = c_3 = 1$ and $b_2 + c_2 \leq 1$. Moreover, the minimal system matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & b_2 & -c_2 \\ 1 & 1 & -1 \end{pmatrix}, \qquad \begin{matrix} b_2, c_2 \in (0, 1], \\ b_2 + c_2 \leq 1, \end{matrix}$$

is a special case of matrix (5.3).

For matrix (5.5), the vector $\gamma_a + \gamma_c = (c_1, 1 - c_2, a_3 - c_3)$ does not violate the minimality of the system S only if $c_1 = 1$. At $c_1 = 1$, other constraints can arise only because of the vectors

$$\gamma_a - \gamma_b + \gamma_c = (0, 1 - b_2 - c_2, a_3 - 1 - c_3) \quad \text{and} \quad 2\gamma_a - \gamma_b + \gamma_c = (0, 2 - b_2 - c_2, 2a_3 - 1 - c_3).$$

The conditions that these vectors do not violate the minimality of S can be written as

$$b_2 = c_2 = 1 \quad \text{or} \quad a_3 \leq c_3$$

and

$$b_2 + c_2 \leq 1 \quad \text{or} \quad 2a_3 \leq c_3,$$

respectively. These constraints lead to matrices of the form (5.3).

For a matrix of the form (5.6), the vector $\gamma_a - \gamma_c = (c_1, 1 - c_2, a_3 - 1)$ does not violate the minimality of the system S only if $c_1 = 1$. But at $c_1 = 1$, the second and third matrices are equivalent, and we again obtain a matrix equivalent to (5.3).

Let us show that the conditions in the statement of the theorem are sufficient. For degenerate systems, this follows from Theorem 8. For matrices of the form (5.3), we always have $\det M_6 \geq 1$. Indeed, if $a_3 \leq c_3/2$, then

$$\det M_6 \geq 1 + c_3\left(1 - \frac{b_2 + c_2}{2}\right) \geq 1,$$

and if $a_3 \leq c_3$ and $b_2 + c_2 \leq 1$, then

$$\det M_6 \geq 1 + c_3(1 - b_2 - c_2) \geq 1.$$

Therefore, the system determined by matrix (5.3) is nondegenerate, and by Theorem 7, the necessary minimality conditions verified above are also sufficient.

5.2. Minimal Systems without Diagonal Dominance. Theorem 9 gives a classification of all minimal systems lying in two octants. For this reason, in what follows, we consider only those triples whose vectors belong to three pairwise different octants.

Theorem 10. *Let $S = (\gamma_a, \gamma_b, \gamma_c)$ be a system of vectors in a lattice Γ such that γ_a, γ_b, and γ_c belong to three different octants. If the system S is a minimal system without diagonal dominance, then the matrix $M(\gamma_a, \gamma_b, \gamma_c)$ can be reduced to one of the following two forms by elementary transformations (see Fig.* 3*):*

$$M_7 = \begin{pmatrix} 1 & 1 & c_1 \\ a_2 & -b_2 & 1 \\ -a_3 & -b_3 & 1 \end{pmatrix}, \qquad \begin{array}{c} \textit{either} \quad c_1 = 1, \quad 2a_2 + b_2 \geq 2; \\ c_1 = 1, \quad a_2 + b_2 \geq 1, \quad b_3 \leq 2a_3; \\ \textit{or} \quad a_2 + b_2 \geq 1, \quad b_3 \leq a_3, \end{array} \tag{5.7}$$

and

$$M_8 = \begin{pmatrix} 1 & 1 & c_1 \\ a_2 & -b_2 & 1 \\ -a_3 & b_3 & 1 \end{pmatrix}, \qquad a_2 + b_2 \geq 1, \tag{5.8}$$

where $a_2, a_3, b_2, b_3 \in [0, 1)$ and $c_1 \in [0, 1]$.

In the former case, the system S is a basis in the lattice Γ. In the latter case, the system S can be either a basis or, under the additional conditions

$$c_1 = 0 \qquad \textit{and} \qquad a_3 + b_3 \geq 1, \tag{5.9}$$

a system of index 2 *in the lattice $\Gamma = \langle \gamma_a, \gamma_b, (\gamma_a + \gamma_b + \gamma_c)/2 \rangle$.*

Conversely, matrices (5.7) *and* (5.8) *are not equivalent to matrices with diagonal dominance; the system $S = (\gamma_a, \gamma_b, \gamma_c)$ with matrix* (5.7) *is a minimal system in the lattice $\Gamma = \langle \gamma_a, \gamma_b, \gamma_c \rangle$; and the system $S = (\gamma_a, \gamma_b, \gamma_c)$ with matrix* (5.8) *is a minimal system in the lattice $\Gamma = \langle \gamma_a, \gamma_b, \gamma_c \rangle$. Moreover, under the additional conditions* (5.9)*, the last system is also a minimal system of index* 2 *in the lattice $\Gamma = \langle \gamma_a, \gamma_b, (\gamma_a + \gamma_b + \gamma_c)/2 \rangle$.*

Proof. Let us prove the necessity of the conditions in the statement of the theorem. The matrix of the system $S = (\gamma_a, \gamma_b, \gamma_c)$ cannot be reduced to a matrix with diagonal dominance if and only if two points of the system S (e.g., γ_a and γ_b) belong to the interior of a face of the parallelepiped $\Pi(\gamma_a, \gamma_b, \gamma_c)$ and the

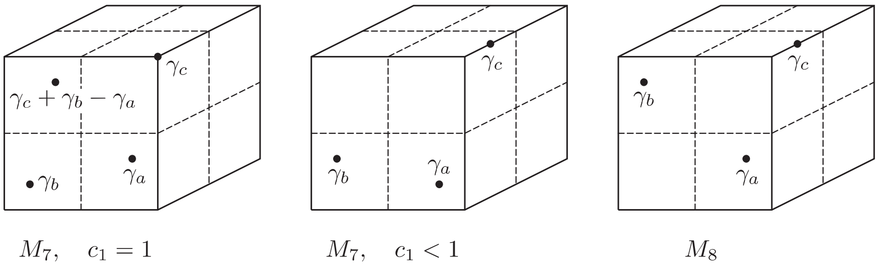

Fig. 3. The arrangement of the points of minimal systems with matrices M_7 and M_8

third point (γ_c in the case under consideration) belongs to the edge perpendicular to this face. Therefore, it is sufficient to consider only minimal systems $S=(\gamma_a,\gamma_b,\gamma_c)$ with matrices of the form

$$\begin{pmatrix} 1 & 1 & c_1 \\ \alpha_2 a_2 & \beta_2 b_2 & 1 \\ \alpha_3 a_3 & \beta_3 b_3 & 1 \end{pmatrix}, \qquad \begin{gathered} 0\le c_1\le 1, \\ 0\le a_2,a_3,b_2,b_3<1, \\ \alpha_2,\alpha_3,\beta_2,\beta_3=\pm 1. \end{gathered}$$

First, note that $(\alpha_2,\alpha_3)\neq(\beta_2,\beta_3)$, because otherwise the vector $\gamma_a-\gamma_b=(0,\alpha_2(a_2-b_2),\alpha_3\times(a_3-b_3))$ violates the minimality of the system S. Taking into account the possibility of interchanging the first two columns and the last two rows, we obtain the four possible arrangements of signs:

$$\begin{pmatrix} \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{pmatrix}=\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

In the first case, we have $a_2a_3b_2b_3=0$, because otherwise we arrive at a contradiction to the assumption that γ_a, γ_b, and γ_c belong to three different octants. Changing the sign assigned to the zero element, we reduce the first case to the second or the third.

In the second case, the vector

$$\gamma_a-\gamma_c=(1-c_1,a_2-1,a_3-1)$$

does not violate the minimality of the system S only if $c_1a_2a_3=0$. If $c_1=0$, then the second case can be reduced to the third by changing signs in the rows and columns of the matrix $M(\gamma_a,\gamma_b,\gamma_c)$, and if $a_2=0$, then it can be reduced to the fourth case. The equality $a_3=0$ cannot hold, because at $a_3=0$, the vector $\gamma_a-\gamma_b=(0,a_2-b_2,b_3)$ violates the minimality of the system S.

In the two remaining cases, degeneracy is verified by using Theorem 8. For nondegenerate systems, we obtain necessary conditions on vector coordinates by using Theorem 7. For triples $(\gamma_a,\gamma_b,\gamma_c)$ being bases, the found necessary conditions on the coefficients are also sufficient.

Consider the case where

$$\begin{pmatrix} \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{pmatrix}=\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}.$$

Since the linear combinations of the vectors γ_a, γ_b, and γ_c with coefficients (4.5) do not vanish, it follows by Theorem 8 that the system S is nondegenerate. The minimality of the system S may be violated by the vectors

$$\begin{aligned} \gamma_a-\gamma_b&=(0,a_2+b_2,-a_3+b_3), \\ \gamma_b-\gamma_a+\gamma_c&=(c_1,-a_2-b_2+1,a_3-b_3+1), \end{aligned}$$

and

$$2\gamma_a-\gamma_b-\gamma_c=(1-c_1,2a_2+b_2-1,-2a_3+b_3-1).$$

The conditions that these vectors do not belong to the interior of the cube $\Pi = (-1,1)^3$ can be written as

$$a_2 + b_2 \geq 1,$$
$$c_1 = 1 \quad \text{or} \quad b_3 \leq a_3 \quad (\text{or} \quad a_2 = b_2 = 0),$$

and

$$\text{either} \quad \text{(i)}\, c_1 = 0 \text{ or } b_3 \leq 2a_3 \quad \text{or} \quad \text{(ii)}\, 2a_2 + b_2 \geq 2 \quad (\text{or} \quad a_2 = b_2 = 0),$$

respectively (the conditions contradicting the inequality $a_2 + b_2 \geq 1$ are parenthesized). Thus, the matrix $M(\gamma_a, \gamma_b, \gamma_c)$ must have the form (5.7).

Consider the remaining case

$$\begin{pmatrix} \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

In this case, none of the linear combinations of the vectors γ_a, γ_b, and γ_c with coefficients (4.5) vanishes, and by Theorem 8, the system S is nondegenerate. Among the linear combinations of γ_a, γ_b, and γ_c with coefficients (4.2) only $\gamma_a - \gamma_b = (0, a_2 + b_2, a_3 + b_3)$ may violate the minimality of the system S. The condition $\gamma_a - \gamma_b \notin \Pi$ is equivalent to the fulfillment of one of the two equivalent inequalities $a_2 + b_2 \geq 1$ and $a_3 + b_3 \geq 1$. Without loss of generality, we can assume that $a_2 + b_2 \geq 1$. Under this constraint, the matrix $M(\gamma_a, \gamma_b, \gamma_c)$ takes the form (5.8).

According to Theorem 7, this completely describes necessary minimality conditions in the case where $(\gamma_a, \gamma_b, \gamma_c)$ is a basis of the lattice.

If the set $(\gamma_a, \gamma_b, \gamma_c)$ is not a basis of the lattice, then, by Theorem 5, it forms a system of index 2. Moreover, the linear combinations of the vectors γ_a, γ_b, and γ_c with coefficients (4.3), that is,

$$\gamma_1 = \frac{\gamma_a + \gamma_b + \gamma_c}{2}, \qquad \gamma_2 = \frac{-\gamma_a + \gamma_b + \gamma_c}{2},$$
$$\gamma_3 = \frac{\gamma_a - \gamma_b + \gamma_c}{2}, \qquad \text{and} \qquad \gamma_4 = \frac{\gamma_a + \gamma_b - \gamma_c}{2},$$

must lie outside Π. This cannot be so for matrix (5.7), because the point

$$\gamma_2 = \frac{1}{2}(c_1, 1 - a_2 - b_2, 1 + a_3 - b_3)$$

violates the minimality of S. Suppose that the columns of matrix (5.8) form a system of index 2. The vectors

$$\gamma_1 = \frac{1}{2}(2 - c_1, a_2 - b_2 - 1, -a_3 + b_3 - 1) \qquad \text{and} \qquad \gamma_2 = \frac{1}{2}(c_1, 1 - a_2 - b_2, 1 + a_3 + b_3)$$

do not belong to Π if and only if condition (5.9) holds. Under these additional constraints, the system $S = (\gamma_a, \gamma_b, \gamma_c)$ can be both a basis in the lattice $\Gamma = \langle \gamma_a, \gamma_b, \gamma_c \rangle$ and a system of index 2 in the lattice $\Gamma = \langle \gamma_a, \gamma_b, (\gamma_a + \gamma_b + \gamma_c)/2 \rangle$.

Let us prove the converse assertion of the theorem. Obviously, none of the systems with matrices (5.7) and (5.8) is a system with diagonal dominance. Let us show that systems with matrices (5.7) and (5.8) are nondegenerate.

For matrices of the form (5.7), we have

$$|\det M_7| = b_2 - b_3 + a_2 + a_3 + c_1(a_2 b_3 + a_3 b_2).$$

The conditions $c_1 = 1$, $b_2 \geq 2 - 2a_2$, $b_3 < 1$, and $a_2 < 1$ imply the estimate

$$|\det M_7| \geq 2 - a_2 - b_3 + 3a_3 + a_2 b_3 - 2a_2 a_3 \geq 1 + \frac{a_2 b_3}{2} \geq 1.$$

If $c_1 = 1$, $b_2 \geq 1 - a_2$, and $a_3 \geq b_3/2$, then

$$|\det M_7| \geq 1 - b_3 + 2a_3 + a_2 b_3 - a_2 a_3 \geq 1 + \frac{a_2 b_3}{2} \geq 1.$$

For $c_1 = 0$, $a_2 + b_2 \geq 1$, and $a_3 \geq b_3$, we have

$$|\det M_7| \geq a_2 + b_2 + a_3 - b_3 \geq 1.$$

For matrices of the form (5.8), the conditions $a_2 + b_2 \geq 1$ and $c_1 a_2 \leq 1$ imply the estimate

$$|\det M_8| = b_2 + b_3 + a_2 + a_3 + c_1(a_3 b_2 - a_2 b_3) \geq 1.$$

Thus, the systems of vectors with matrices (5.7) and (5.8) are nondegenerate. Moreover, for the lattices $\Gamma = \langle \gamma_a, \gamma_b, \gamma_c \rangle$ and $\Gamma = \langle \gamma_a, \gamma_b, (\gamma_a + \gamma_b + \gamma_c)/2 \rangle$ under consideration, the octahedron with vertices $\pm\gamma_a$, $\pm\gamma_b$, and $\pm\gamma_c$ is primitive. Therefore, Theorem 7 applies to the triple of vectors $(\gamma_a, \gamma_b, \gamma_c)$, and the found necessary conditions on matrix elements are also sufficient minimality conditions.

5.3. Minimal Systems of Standard Form. Theorems 9 and 10 describe minimal systems which are contained in two octants or cannot be written in the form of a matrix with diagonal dominance. To complete the classification of minimal systems, it remains to consider the case of systems of the standard form (2.3). As mentioned above, the matrices of such systems can be reduced to one of the canonical forms (recall that to the zero coordinates an arbitrary sign can be assigned)

$$\begin{pmatrix} 1 & b_1 & -c_1 \\ -a_2 & 1 & c_2 \\ a_3 & -b_3 & 1 \end{pmatrix}, \tag{5.10}$$

$$\begin{pmatrix} 1 & b_1 & c_1 \\ -a_2 & 1 & c_2 \\ a_3 & -b_3 & 1 \end{pmatrix}, \tag{5.11}$$

and

$$\begin{pmatrix} 1 & -b_1 & -c_1 \\ -a_2 & 1 & -c_2 \\ -a_3 & -b_3 & 1 \end{pmatrix}, \tag{5.12}$$

where $a_i, b_i, c_i \in [0, 1]$.

Consider separately the exceptional minimal systems of the standard form with indices 2, 0, and 1 (by Theorem 5, there are no other possibilities).

5.4. Exceptional Minimal Systems of the Standard Form of Index 2.

Theorem 11. *Let $S = (\gamma_a, \gamma_b, \gamma_c)$ be a minimal system of the standard form (2.3) having index 2 in the lattice Γ. Then the matrix $M(\gamma_a, \gamma_b, \gamma_c)$ is equivalent to one of the two matrices (see Fig. 4)*

$$M_9 = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ a_3 & -b_3 & 1 \end{pmatrix}, \qquad a_3, b_3 \in [0, 1], \tag{5.13}$$

and

$$M_{10} = \begin{pmatrix} 1 & 1 & 0 \\ -a_2 & 1 & c_2 \\ a_3 & -b_3 & 1 \end{pmatrix}, \qquad \begin{array}{c} a_2, c_2, a_3, b_3 \in [0, 1], \\ a_2 + c_2 \geq 1, \quad a_3 + b_3 \geq 1, \end{array} \tag{5.14}$$

and, moreover, $\Gamma = \langle \gamma_a, \gamma_b, (\gamma_a + \gamma_b + \gamma_c)/2 \rangle$.

Conversely, each of the matrices (5.13) *and* (5.14) *determines a minimal system of index* 2 *in the lattice* $\Gamma = \langle \gamma_a, \gamma_b, (\gamma_a + \gamma_b + \gamma_c)/2 \rangle$.

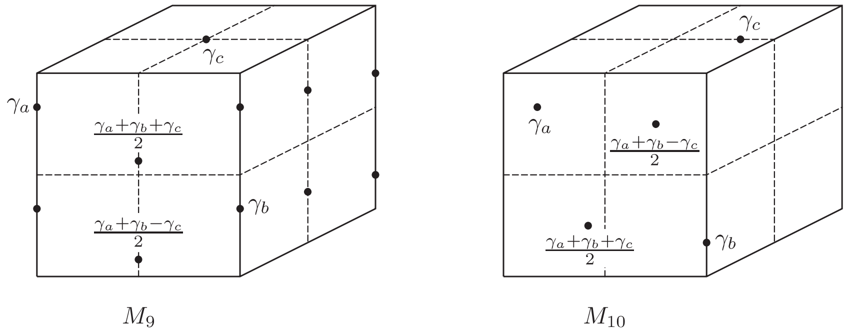

Fig. 4. The arrangement of the points of minimal systems with matrices M_9 and M_{10}

Proof. By theorem 7, a linearly independent system $S = (\gamma_a, \gamma_b, \gamma_c)$ is a minimal system of index 2 in the lattice $\Gamma = \langle \gamma_a, \gamma_b, (\gamma_a + \gamma_b + \gamma_c)/2 \rangle$ if and only if vectors (5.5.0.5.2) do not belong to the cube $\Pi = (-1, 1)^3$.

For matrices of the form (5.12), we have

$$\gamma_1 = \left(\frac{1 - b_1 - c_1}{2}, \frac{1 - a_2 - c_2}{2}, \frac{1 - a_3 - b_3}{2} \right) \in \Pi;$$

therefore, this arrangement of signs is impossible under the conditions of the theorem.

For matrices of the form (5.10), the vector

$$\gamma_1 = \left(\frac{1 + b_1 - c_1}{2}, \frac{1 - a_2 + c_2}{2}, \frac{1 + a_3 - b_3}{2} \right)$$

does not belong to Π if at least one of the three inequalities

$$b_1 - c_1 \geq 1, \qquad c_2 - a_2 \geq 1, \qquad \text{and} \qquad a_3 - b_3 \geq 1$$

holds. The action of the group G_3 transforms these inequalities into each other. Therefore, without loss of generality, we can assume that the first inequality holds, i.e., $b_1 = 1$ and $c_1 = 0$.

For matrices of the form (5.11), the vector

$$\gamma_4 = \left(\frac{1 + b_1 - c_1}{2}, \frac{1 - a_2 - c_2}{2}, \frac{-1 + a_3 - b_3}{2} \right)$$

does not belong to Π if at least one of the two inequalities

$$b_1 - c_1 \geq 1 \qquad \text{and} \qquad a_3 - b_3 \leq -1$$

holds. These inequalities can be transformed into each other by the action of G_3 too. Therefore, we can again assume the validity of the first inequality, which implies that $b_1 = 1$ and $c_1 = 0$. Thus, in any case, the matrix $M(\gamma_a, \gamma_b, \gamma_c)$ reduces to the form

$$\begin{pmatrix} 1 & 1 & 0 \\ -a_2 & 1 & c_2 \\ a_3 & -b_3 & 1 \end{pmatrix}.$$

For such a matrix, we have $\gamma_1, \gamma_4 \notin \Pi$, and it remains to consider the vectors γ_2 and γ_3. The condition

$$\gamma_3 = \left(0, \frac{-1 - a_2 + c_2}{2}, \frac{1 + a_3 + b_3}{2} \right) \notin \Pi$$

is equivalent to the fulfillment of one of the inequalities

$$c_2 + 1 \le a_2 \qquad \text{and} \qquad a_3 + b_3 \ge 1.$$

If the former inequality holds, then $a_2 = 1$ and $c_2 = 0$. Moreover,

$$\gamma_2 = \left(0, 1, \frac{1 - a_3 - b_3}{2}\right) \notin \Pi,$$

and there arise no other constraints on the vector coordinates. Thus, we obtain the set of matrices of the form (5.13). If $a_3 + b_3 \ge 1$, then the condition

$$\gamma_2 = \left(0, \frac{a_2 + c_2 + 1}{2}, \frac{1 - a_3 - b_3}{2}\right) \notin \Pi$$

is equivalent to the inequality $a_2 + c_2 \ge 1$. In this case, the matrix $M(\gamma_a, \gamma_b, \gamma_c)$ belongs to the set of matrices of the form (5.14).

Let us prove the converse assertion of the theorem. Since $\det M_9 = 2$ and

$$\det M_{10} = 1 + a_2 + c_2(a_3 + b_3) \ge 1 + a_2 + c_2 \ge 2,$$

it follows that each of matrices (5.13) and (5.14) is singular and has index 2 in the lattice $\Gamma = \langle \gamma_a, \gamma_b, (\gamma_a + \gamma_b + \gamma_c)/2\rangle$. By Theorem 7, the necessary minimality condition verified above are also sufficient.

5.5. Degenerate Exceptional Minimal Systems of the Standard Form.

Theorem 12. *Let $S = (\gamma_a, \gamma_b, \gamma_c)$ be a degenerate minimal system of the standard form in a lattice Γ. Then the matrix $M(\gamma_a, \gamma_b, \gamma_c)$ is equivalent to the matrix (see Fig. 5)*

$$M_{11} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ \frac{b_3-1}{2} & -b_3 & 1 \end{pmatrix}, \qquad b_3 \in [0, 1]. \tag{5.15}$$

Conversely, any system of vectors whose matrix can be reduced to the form (5.15) *is a degenerate minimal system of the standard form in the lattice* Γ.

The converse is also true: any system of three vectors $(\gamma_a, \gamma_b, \gamma_c)$ whose matrix can be reduced to the form (5.15) *is a minimal system in the lattice (of rank* 2*)* $\Gamma = \langle \gamma_a, \gamma_b, \gamma_c\rangle = \langle \gamma_b, \gamma_c\rangle$.

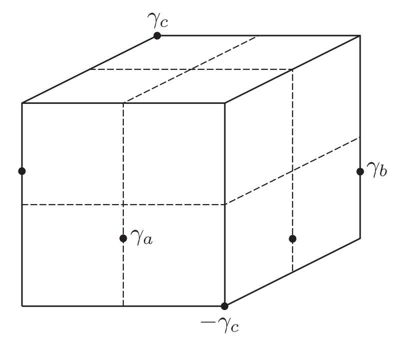

Fig. 5. The arrangement of the points of the minimal system with matrix M_{11}

Proof. If the matrix of a system S has the form (5.10), then among the linear combinations of the vectors γ_a, γ_b, and γ_c with coefficients (4.5) only the vector $\gamma_a + \gamma_b + \gamma_c = (1 + b_1 - c_1, 1 - a_2 + c_2, 1 + a_3 - b_3)$ can vanish. Moreover, this can happen only if $a_3 = b_1 = c_2 = 0$ and $a_2 = b_3 = c_1 = 1$. But in this case, the given minimal system is not exceptional, because it is a special case of the minimal system (2.6).

If the matrix $M(\gamma_a,\gamma_b,\gamma_c)$ has the form (5.11), then among the linear combinations of the vectors γ_a, γ_b, and γ_c with coefficients (4.5) only the vectors

$$\gamma_a+\gamma_b-\gamma_c=(1+b_1-c_1,1-a_2-c_2,a_3-b_3-1)$$

and

$$\gamma_a+2\gamma_b-\gamma_c=(1+2b_1-c_1,2-a_2-c_2,a_3-2b_3-1)$$

can vanish. If $\gamma_a+\gamma_b-\gamma_c=0$, then $b_1=b_3=0$, $c_1=a_3=0$, and $a_2+c_2=1$, and the matrix

$$M(\gamma_a,\gamma_b,\gamma_c)=\begin{pmatrix}1&0&1\\-a_2&1&1-a_2\\1&0&1\end{pmatrix}$$

is equivalent to matrix (2.6). Thus, the system S is not exceptional in this case too. If $\gamma_a+2\gamma_b+\gamma_c=0$, then $b_1=b_3=0$, $c_1=a_3=0$, and $a_2=c_2=1$, and the matrix

$$M(\gamma_a,\gamma_b,\gamma_c)=\begin{pmatrix}1&0&1\\-1&1&1\\1&0&1\end{pmatrix}$$

is equivalent to matrix (5.15) with $b_3=1$.

Now, suppose that a system with matrix of the form (5.12) is exceptional. Then, by the definition of an exceptional system, the equality $\gamma_a+\gamma_b+\gamma_c=0$ cannot hold. Among the linear combinations of the vectors γ_a, γ_b, and γ_c with coefficients (4.5) only one of the vectors

$$2\gamma_a+\gamma_b+\gamma_c,\qquad \gamma_a+2\gamma_b+\gamma_c,\qquad\text{and}\qquad \gamma_a+\gamma_b+2\gamma_c$$

can vanish. Since the situation is symmetric, we can assume that the first vector vanishes. Then $b_1=c_1=1$, $2a_2+c_2=1$, and $2a_3+c_3=1$. The linear combinations of γ_a, γ_b, and γ_c with coefficients (4.4) because of which additional constraints on coordinates arise are

$$\gamma_a+\gamma_b=(0,1-a_2,-a_3-b_3)\qquad\text{and}\qquad \gamma_a+\gamma_c=(0,c_2-a_2,1-a_3).$$

The requirements $\gamma_a+\gamma_b$ and $\gamma_a+\gamma_c\notin\Pi$ are equivalent to the conditions

$$a_2=0\quad\text{or}\quad a_3+b_3\geq 1$$

and

$$\text{either}\quad (\mathrm{i})\ c_2=1 \text{ and } a_2=0\quad\text{or}\quad c_2=0\text{ and } a_2=1\qquad\text{or}\qquad (\mathrm{ii})\ a_3=0,$$

respectively. Thus, we have two types of solutions:

$$b_1=c_1=c_2=1,\qquad a_2=0,\qquad 2a_3+b_3=1$$

and

$$b_1=c_1=b_3=1,\qquad a_3=0,\qquad 2a_2+b_2=1.$$

The corresponding matrices

$$\begin{pmatrix}1&-1&-1\\0&1&-1\\\frac{b_3-1}{2}&-b_3&1\end{pmatrix}\qquad\text{and}\qquad\begin{pmatrix}1&-1&-1\\\frac{c_2-1}{2}&1&-c_2\\0&-1&1\end{pmatrix}$$

are both equivalent to matrix (5.15).

Let us prove the converse assertion of the theorem. The system of vectors with matrix (5.15) is degenerate ($2\gamma_a+\gamma_b+\gamma_c=0$) and generates the lattice $\Gamma=\langle\gamma_b,\gamma_c\rangle$. By Theorem 8, the necessary minimality condition verified above are also sufficient.

5.6. Exceptional Minimal Bases of the Standard Form.

Theorem 13. *Let $S=(\gamma_a,\gamma_b,\gamma_c)$ be a minimal system of the standard form in a lattice Γ. If S is a basis in Γ, then the matrix $M(\gamma_a,\gamma_b,\gamma_c)$ is equivalent to the matrix (see Fig. 6)*

$$M_{12}=\begin{pmatrix} 1 & -1 & -1 \\ -a_2 & 1 & -c_2 \\ -a_3 & -b_3 & 1 \end{pmatrix}, \qquad \begin{gathered} a_2,a_3,b_3,c_2\in[0,1], \\ 2a_2+b_2\ge 2, \quad a_3+b_3\ge 1. \end{gathered} \tag{5.16}$$

Conversely, any system with matrix equivalent to (5.16) *is a minimal system in the lattice* $\Gamma=\langle\gamma_a,\gamma_b,\gamma_c\rangle$.

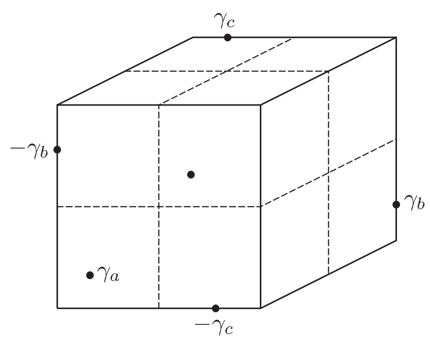

Fig. 6. The arrangement of the points of the minimal system with matrix M_{12}

Proof. As mentioned above, the matrix $M(\gamma_a,\gamma_b,\gamma_c)$ can be reduced to one of the canonical forms (5.10)–(5.12) by elementary transformations. By the definition of Minkowski bases, a minimal system being a basis can be exceptional only if its matrix is equivalent to matrix (5.12).

As above, to find necessary and sufficient conditions on the elements of the matrix $M(S)$, consider linear combinations of the vectors γ_a, γ_b, and γ_c with coefficients (4.2). The condition

$$\gamma_a+\gamma_b+\gamma_c=(1-b_1-c_1,1-a_2-c_2,1-a_3-b_3)\notin\Pi$$

is equivalent to the fulfillment of one of the inequalities

$$|1-b_1-c_1|\ge 1, \qquad |1-a_2-c_2|\ge 1, \qquad \text{and} \qquad |1-a_3-b_3|\ge 1.$$

Since these inequalities are equivalent, we can assume that the first inequality holds. This is possible if $b_1=c_1=0$ or $b_1=c_1=1$. The former case cannot occur, because at $b_1=c_1=0$, the system S is not exceptional; indeed, assigning positive signs to the coefficients b_1 and c_1, we obtain the matrix of a Minkowski basis of type II (which corresponds to the signature 13 in Table 1). Therefore, we can assume in what follows that $b_1=c_1=1$. Among the linear combinations of the vectors γ_a, γ_b, and γ_c with coefficients (4.2), only the vectors $\gamma_a+\gamma_b$, $\gamma_a+\gamma_c$, and $2\gamma_a+\gamma_b+\gamma_c$ may violate the minimality of the system S. Consider the vector

$$2\gamma_a+\gamma_b+\gamma_c=(0,1-2a_2-2c_2,1-2a_3-2c_3).$$

It does violate the minimality of S if and only if at least one of the following four conditions holds:

$$a_2=c_2=0, \qquad a_3=b_3=0, \qquad 2a_2+c_2\ge 2, \qquad \text{and} \qquad 2a_3+b_3\ge 2.$$

The first two conditions are equivalent to the last two conditions. Therefore, it is sufficient to consider two cases, $a_2=c_2=0$ and $2a_2+c_2\ge 2$. The former case cannot occur, because if $a_2=c_2=0$, then the system S is a Minkowski basis and, hence, is not exceptional. In the latter case, $\gamma_a+\gamma_c\notin\Pi$, and the condition $\gamma_a+\gamma_b=(0,1-a_2,-a_3-b_3)\notin\Pi$ is equivalent to the inequality $a_3+b_3\ge 1$. Thus, the matrix $M(\gamma_a,\gamma_b,\gamma_c)$ must have the form (5.16).

Suppose that the system S has matrix (5.16). Then S is nondegenerate. Indeed,

$$-\det M_{12} = -1 + a_2 + a_3 + c_2 b_3 + c_2 a_3 + b_3 a_2.$$

Using the inequalities $c_2 \geq 2 - 2a_2$ and $b_3 \geq 1 - a_3$, we obtain the estimate

$$-\det M_{12} \geq 1 + a_3(1 - a_2) \geq 1.$$

Therefore, by Theorem 7, the minimality conditions verified above are also sufficient.

6. A THREE-DIMENSIONAL ANALOGUE OF VAHLEN'S THEOREM FOR REDUCIBLE LATTICES

Using the classification of minimal systems given by Theorems 9–13, we can extend the three-dimensional analogue of Vahlen's theorem (1.3) to arbitrary lattices as follows.

Theorem 14. *If $S = (\gamma_a, \gamma_b, \gamma_c)$ is a completely minimal system in a lattice Γ and the matrix $M(\gamma_a, \gamma_b, \gamma_c)$ has the form* (2.2)*, then*

$$a_1 a_2 a_3 + b_1 b_2 b_3 + c_1 c_2 c_3 \leq \det \Gamma. \tag{6.1}$$

For an arbitrary minimal system $S = (\gamma_a, \gamma_b, \gamma_c)$,

$$a_1 a_2 a_3 + b_1 b_2 b_3 + c_1 c_2 c_3 \leq 2 \det \Gamma. \tag{6.2}$$

Remark 2. If $S = (\gamma_a, \gamma_b, \gamma_c)$ is a minimal system in a lattice Γ, then, by Minkowski's convex body theorem, we have $\mathrm{vol}(\Pi(S)) \leq \det \Gamma$. Moreover,

$$\max\{a_1 a_2 a_3, b_1 b_2 b_3, c_1 c_2 c_3\} \leq \mathrm{vol}(\Pi(S)).$$

Therefore, the trivial estimate (1.4) always holds, and Theorem 14 can be regarded as a sharpening of this estimate.

Proof (of Theorem 14)**.** For each matrix M written in the form (2.2), we define the function

$$\Delta_k(M) = k \det \Gamma - a_1 a_2 a_3 - b_1 b_2 b_3 - c_1 c_2 c_3, \qquad k = 1, 2.$$

For matrices of the standard form (2.3), the functions $\Delta_k(M)$ have the form

$$\Delta_k(M) = k \det \Gamma - a_2 a_3 - b_1 b_3 - c_1 c_2, \qquad k = 1, 2.$$

Let us prove that, for minimal triples, we have $\Delta_2(M) \geq 0$, and for completely minimal triples, $\Delta_1(M) \geq 0$.

For minimal Minkowski systems (2.4)–(2.6), the inequality $\Delta_1(M) \geq 0$ can be proved in precisely the same way as in the case of irreducible lattices (see [21]). Indeed, for matrices of the form (2.4), inequalities $b_1 \leq 1$ and $b_1 \leq c_1$ imply the estimates

$$\begin{aligned}
\Delta_1(M_1) &= 1 + c_2 b_3 - a_2 a_3 + b_1(a_2 + c_2 a_3 - b_3) - c_1(c_2 + a_2 b_3 - a_3) \\
&\geq b_1(1 + c_2 b_3 - a_2 a_3 + a_2 + c_2 a_3 - b_3) - c_1(c_2 + a_2 b_3 - a_3) \\
&\geq c_1(1 + c_2 b_3 - a_2 a_3 + a_2 + c_2 a_3 - b_3 - c_2 - a_2 b_3 + a_3) \\
&= c_1(a_3(1 - a_2) + a_2(1 - b_3) + (1 - c_2)(1 - b_3) + c_2 a_3) \geq 0.
\end{aligned}$$

Similarly, for matrices of the form (2.5), the inequalities $b_1 \leq 1$, $b_1 \leq c_1$, and $a_2 + c_2 \geq 1$ imply

$$\begin{aligned}
\Delta_1(M_2) &= 1 + c_2 b_3 - a_2 a_3 + b_1(a_2 + c_2 a_3 - b_3) + c_1(a_2 b_3 - a_3 - c_2) \\
&\geq b_1(1 + c_2 b_3 - a_2 a_3 + a_2 + c_2 a_3 - b_3) + c_1(a_2 b_3 - a_3 - c_2) \\
&\geq c_1(1 + c_2 b_3 - a_2 a_3 + a_2 + c_2 a_3 - b_3 + a_2 b_3 - a_3 - c_2) \\
&= c_1((1 - c_2)(1 - a_3) + a_2(1 - a_3) + b_3(a_2 + c_2 - 1)) \geq 0.
\end{aligned}$$

For a minimal system with matrix (2.6), we have

$$a_2 a_3 + b_1 b_3 + c_1 c_2 \leq (b_1 + c_1)(a_2 + c_2)(a_3 + b_3) = 1.$$

It follows from Minkowski's convex body theorem (see Remark 2) that $\det\Gamma \ge 1$. Therefore, $a_2a_3 + b_1b_3 + c_1c_2 \le 1 \le \det\Gamma$ and $\Delta_1(M_3) \ge 0$.

We proceed to analyze exceptional minimal systems. First, consider minimal systems lying in two octants. Again applying the inequality $\det\Gamma \ge 1$, we obtain the following estimate for a matrix of the form (5.1):

$$\Delta_1(M_4) \ge 1 - b_2b_3 - (1-b_2)(1-b_3) = b_2 + b_3 - b_2b_3 \ge 0.$$

Similarly, for a matrix of the form (5.2), we have

$$\Delta_2(M_5) = 2\det\Gamma - 1 - c_3 \ge 1 - c_3 \ge 0.$$

A system with matrix of the form (5.2) cannot be completely minimal, because γ_a violates the minimality of γ_b.

For a matrix of the form (5.3), we have

$$\Delta_2(M_6) = 2 + 2c_3 - 2a_3(b_2 + c_2) - b_2 - c_2c_3.$$

If $a_3 \le c_3/2$, then

$$\Delta_2(M_6) \ge c_3(2 - b_2 - c_2) + 2 - b_2 - c_2c_3 \ge 0,$$

and if $a_3 \le c_3$ and $c_2 \le 1 - b_2$, then

$$\Delta_2(M_6) \ge 2 - b_2 - c_3 + b_2c_3 \ge 1.$$

Suppose that matrix (5.3) determines a completely minimal system. Then $c_2 \ne 1$ (otherwise, γ_a violates the minimality of γ_c). Therefore, the vector

$$\gamma_a - \gamma_b + \gamma_c = (0, 1 - b_2 - c_2, a_3 - 1 - c_3)$$

does not violate the minimality of the node $\gamma_b = (1, b_2, 1)$ only if $b_2 \le (1 - c_2)/2$. Under this constraint, the inequality $a_3 \le c_3$ implies

$$\Delta_1(M_6) \ge \frac{1}{2}(1 + c_2 + c_3 - 3c_2c_3) \ge 0.$$

Now, consider minimal systems without diagonal dominance. If $c_1 = 1$ in matrix (5.7), then the system $(\gamma_a, \gamma_b, \gamma_c)$ is not completely minimal (γ_a and γ_b violate the minimality of γ_c), and it suffices to prove inequality (6.2).

Suppose that $2a_2 + b_2 \ge 2$. In the representation

$$\Delta_2(M_7) = 2(a_2 + a_3 + b_2 - b_3 + a_2b_3 + a_3b_2) - a_2a_3 - b_2b_3 - 1, \tag{6.3}$$

the coefficient of b_2 is positive and the coefficient of b_3 is negative. Hence, it is sufficient to check the estimate $\Delta_2(M_7) \ge 0$ for $b_2 = 2 - 2a_2$ and $b_3 = 1$. In this case,

$$\Delta_2(M_7) = 2a_2 - 1 + 6a_3 - 5a_2a_3 \ge 2a_2 - 1 \ge 0,$$

because the inequality $2a_2 + b_2 \ge 2$ can hold only if $a_2 \ge 1/2$.

Suppose that $a_2 + b_2 \ge 1$ and $a_3 \ge b_3/2$. The coefficient of a_3 in (6.3) is negative, and, as mentioned above, the coefficient of b_2 is positive. Therefore, it suffices to check the estimate $\Delta_2(M_7) \ge 0$ for $b_2 = 1 - a_2$ and $a_3 = b_3/2$. For these values of b_2 and a_3, we have

$$\Delta_2(M_7) = 1 - b_3 + \frac{3}{2}a_2b_3 \ge 1 - b_3 \ge 0.$$

Consider the case where the elements of matrix (5.7) are related by the inequalities $a_2 + b_2 \ge 1$ and $b_3 \le a_3$. In the decomposition

$$\Delta_1(M_7) = a_2 + a_3 + b_2 - b_3 + c_1(a_2b_3 + a_3b_2 - 1) - a_2a_3 - b_2b_3,$$

the coefficient of b_2 is positive, and the coefficient of b_3 is negative. Therefore, it suffices to verify the estimate $\Delta_1(M_7) \ge 0$ for $b_2 = 1 - a_2$ and $b_3 = a_3$. In this case, we have

$$\Delta_1(M_7) = (1 - a_3)(1 - c_1) \ge 0.$$

If a minimal system with matrix (5.8) is a basis in the lattice, then $|\det M_8| = \det\Gamma$ and

$$\Delta_1(M_8) = a_2 + a_3 + b_2 + b_3 + c_1(a_3b_2 - a_2b_3) - a_2a_3 - b_2b_3 - c_1.$$

Here, the coefficient of b_2 is positive and the coefficient of c_1 is negative. Therefore, it is sufficient to prove the estimate $\Delta_1(M_8) \geq 0$ for $c_1 = 1$ and $b_2 = 1 - a_2$. In this case, we have

$$\Delta_1(M_8) = 2a_3(1 - a_2) \geq 0.$$

If the minimal system has index 2, then $|\det M_8| = 2\det\Gamma$ and the additional condition $c_1 = 0$ holds. Hence,

$$\Delta_1(M_8) = \frac{1}{2}|\det M_5| - a_2a_3 - b_2b_3 = \frac{1}{2}(a_2 + a_3 + b_2 + b_3) - a_2a_3 - b_2b_3 \geq 0,$$

because $b_2 \geq b_2b_3$, $b_3 \geq b_2b_3$, $a_2 \geq a_2a_3$, and $a_3 \geq a_2a_3$.

To analyze minimal systems of index 2, we use their classification from Theorem 11. For matrices of the form (5.13), we have

$$\det M_9 = 2 = 2\det\Gamma \qquad \text{and} \qquad \Delta_2(M_9) = 2 - a_3 - b_3 \geq 0.$$

If matrix (5.13) determines a completely minimal system, then $a_3 = b_3$ (because the nodes γ_a and γ_b must not violate the minimality of each other). The vector

$$\frac{1}{2}(\gamma_a - \gamma_b - \gamma_c) = \left(0, -1, a_3 - \frac{1}{2}\right)$$

must not violate the minimality of the nodes γ_a and γ_b either. This is possible only if $a_3 - 1/2 < -a_3$, i.e., if $a_3 < 1/4$. Under this constraint,

$$\Delta_1(M_9) = 1 - a_3 - b_3 = 1 - 2a_3 > \frac{1}{2} > 0.$$

For matrices of the form (5.14), $\det M_{10} = 2\det\Gamma$ also. Therefore,

$$\Delta_2(M_{10}) = c_2b_3 + 1 - a_2a_3 - b_3 + a_2 + a_3c_2 = (a_2 + c_2)(1 - a_3) + (1 - c_2)(1 - b_3) + 2a_3c_2 \geq 0.$$

Matrix (5.14) cannot determine a completely minimal system, because if $a_3 + b_3 \geq 1$, then the vector

$$\frac{1}{2}(\gamma_a + \gamma_b - \gamma_c) = \left(1, \frac{1 - a_2 - c_2}{2}, \frac{-1 + a_3 - b_3}{2}\right)$$

violates the minimality of the node $\gamma_b = (1, 1, b_3)$.

For matrices of the form (5.15), the assertion of the theorem is proved in the same way as for matrices of the form (5.2).

Now, consider exceptional bases, which are classified by Theorem 13. Estimate (6.2) follows from the relations

$$\Delta_2(M_{12}) = (2 + c_2 - a_2)(a_3 + b_3 - 1) + 2b_3(2a_2 + c_2 - 2) + b_3(1 - c_2) + (a_2 + c_2)a_3 \geq 0.$$

Suppose that matrix (5.16) determines a completely minimal system. The vector

$$\gamma_d = \gamma_a + \gamma_b + \gamma_c = (-1, 1 - a_2 - c_2, 1 - a_3 - b_3)$$

must not violate the minimality of the node $\gamma_b = (-1, 1, -b_3)$. This is possible if at least one of the two equivalent conditions $a_2 = 1$ and $a_3 = 1$ holds. Without loss of generality, we can assume that $a_2 = 1$. Since γ_a and γ_b do not violate the minimality of each other, it follows that $a_3 = b_3$ (moreover, $a_3 + b_3 \geq 1$, i.e., $a_3 = b_3 > 0$). The node γ_c does not violate the minimality of γ_a; therefore, $c_2 = 1$. Finally, the node $\gamma_d = (-1, -1, c_3 - 2a_3)$ must not violate the minimality of $\gamma_c = (-1, -1, 1)$. Therefore, $a_3 = 1$, and the matrix $M(\gamma_a, \gamma_b, \gamma_c)$ takes the form

$$\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

For the minimal system with such matrix, $\Delta_1(M_{12}) = 1$.

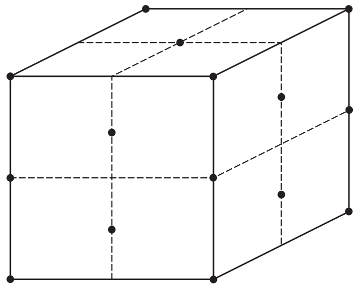

Fig. 7

Remark 3. The constant 2 in estimate (6.2) is unimprovable. The elements of the matrices M_j ($j = 5, 6, 7, 9, 10, 11, 12$) can be chosen so that the relation $\Delta_2(M_j) = 0$ holds (in the limit for the matrix M_7). In all of the cases, the corresponding extremal lattice has basis matrix (up to equivalent transformations)

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & -1 \\ \frac{1}{2} & 1 & -1 \end{pmatrix}.$$

Figure 7 shows the points of the extremal lattice on the surface of the cube Π.

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