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Buchstaber Formal Group and Elliptic Functions of Small Levels

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Abstract—In the paper, we suggest a method for finding relations concerning series defining the Buchstaber formal group. This method is applied to the cases in which the exponent of the group is an elliptic function of level $n = 2, 3$, and 4. An algebraic relation for the series defining the universal Buchstaber formal group is also proved.

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1. INTRODUCTION

We follow the standard notation for the Weierstrass σ-, ζ-, and $\wp$-functions constructed from the lattice $\Gamma = 2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z}$ with periods $2\omega_1$, $2\omega_2$, $\operatorname{Im}\omega_2/\omega_1 > 0$ (see, e.g., [1]). The lattice Γ is always assumed to be fixed and, therefore, the dependence of functions on Γ is not indicated as a rule in what follows.

In [2], for the Baker–Akhiezer function

$$\Phi(x) = \frac{\sigma(z-x)}{\sigma(x)\sigma(z)}\, e^{\zeta(z)x},$$

the following additivity theorem was proved:

$$\Phi(x+y) = \frac{\Phi(x)\Phi'(y) - \Phi'(x)\Phi(y)}{\wp(x) - \wp(y)}\,.$$

It implies that, for an arbitrary α, the function

$$f(x) = \frac{e^{\alpha x}}{\Phi(x)} = \frac{\sigma(x)\sigma(z)}{\sigma(z-x)}\, e^{\alpha x - \zeta(z)x}, \tag{1.1}$$

where $z \not\equiv 0 \pmod{\Gamma}$, satisfies the relation

$$f(x+y) = \frac{f(x)^2 a(y) - f(y)^2 a(x)}{f(x)b(y) - f(y)b(x)}\,, \tag{1.2}$$

in which $a(x) = f(x)^2(\wp(x) + \lambda)$,

$$b(x) = f'(x) + \mu f(x), \tag{1.3}$$

and the parameters λ and μ can be chosen arbitrarily. The simplest formulas are obtained for $\lambda = -\wp(z)$ and, therefore, everywhere below we shall assume that

$$a(x) = f(x)^2(\wp(x) - \wp(z)). \tag{1.4}$$

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Formula (1.2) means that the function (1.1) is the exponential of the Buchstaber formal group

$$F(u,v)=\frac{u^2A(v)-v^2A(u)}{uB(v)-vB(u)}\,, \tag{1.5}$$

where

$$A(t)=1+\sum_{i=1}^{\infty}A_it^i,\qquad B(t)=1+\sum_{i=1}^{\infty}B_it^i.$$

The corresponding passage is carried out by the changes

$$u=f(x),\qquad v=f(y),\qquad A(t)=a(f^{-1}(t)),\qquad B(t)=b(f^{-1}(t)).$$

It follows from the form of the group (1.5) that the coefficients A_2 and B_1 of the series $A(t)$ and $B(t)$ can be chosen arbitrarily.

The converse problem was proved in [3]; namely, it was proved that the exponential of the formal group law (1.5) is a function of the form (1.1). Let us present the precise formulation of this result from the book [4] (see Theorem E 5.4).

Theorem 1. *Let a formal group law* (1.5) *be given, where $A(t)$, $B(t)\in R[[t]]$, let R be a torsion-free ring, and let $A(0)=B(0)=1$. Then the exponential of the law* (1.5) *is of the form* (1.1)*, where α and the parameters $\wp(z)$, $\wp'(z)$, and g_2 defining the Baker–Akhiezer function $\Phi(x,z)$ can be expressed using the coefficients of the series $A(t)$ and $B(t)$ using the expansion*

$$\begin{aligned}f(x)&=x+2\alpha\frac{x^2}{2}+3(\alpha^2+\wp(z))\frac{x^3}{3!}+4(\alpha^3-\wp'(z)+3\alpha\wp(z))\frac{x^4}{4!}\\&\quad+(5\alpha^4+30\alpha^2\wp(z)+45\wp(z)^2-20\alpha\wp'(z)-3g_2)\frac{x^5}{5!}+\cdots.\end{aligned}$$

The subsequent considerations use this theorem; therefore, everywhere below, it is assumed that the formal group laws are considered over torsion-free rings.

For a positive integer $n>1$, denote by z_n the point of exact order n on the torus $\mathbb{C}/\Gamma$. In other words, $z_n=(2\omega_1k+2\omega_2l)/n$, where k and l are integers and $(k,l,n)=1$. Following [5]–[7], we define an *elliptic function of level n* by the relation

$$f_n(x)=\frac{\sigma(x)\sigma(z_n)}{\sigma(z_n-x)}\,e^{-h_nx}, \tag{1.6}$$

where $h_n=(2\eta_1k+2\eta_2l)/n$ ($\eta_j=\zeta(\omega_j)$, $j=1,2$). Every function of this kind is a special case of the exponential of the Buchstaber formal group and is obtained by substituting $z=z_n$ and

$$\alpha=\alpha_n=\zeta(z_n)-h_n \tag{1.7}$$

into relation (1.6). The series $a(x)$ and $b(x)$ defining the addition law (1.2) become

$$a_n(x)=f_n(x)^2(\wp(x)-\wp(z_n)), \tag{1.8}$$

$$b_n(x)=f_n'(x)+\mu_nf_n(x), \tag{1.9}$$

where μ_n can be chosen arbitrarily. Here, in the formal group law (1.5), the following functions are involved:

$$A_n(t)=a_n(f_n^{-1}(t)),\qquad B_n(t)=b_n(f_n^{-1}(t)),$$

i.e., some specialization of the group (1.5) is obtained in every case.

The following three functions, which can be expressed using the Jacobi elliptic functions, are elliptic functions of level 2:

- $f(x)=\operatorname{sn}x$ (for $z=\omega_2$);
- $f(x)=\operatorname{sc}x:=\operatorname{sn}x/\operatorname{cn}x$ (for $z=\omega_1$);

- $f(x) = \operatorname{sd} x := \operatorname{sn} x / \operatorname{dn} x$ (for $z = \omega_1 + \omega_2$).

They satisfy the additivity theorem

$$f(u+v) = \frac{f(u)^2 - f(v)^2}{f(u)f'(v) - f(v)f'(u)},$$

to which the following formal group law corresponds:

$$F(u,v) = \frac{u^2 - v^2}{uB(v) - vB(u)}. \tag{1.10}$$

Moreover, as is well known, the Jacobi elliptic sine $y(z) = \operatorname{sn}(z,k)$ is uniquely defined as the solution of the differential relation

$$y'(z)^2 = (1 - \xi y(z)^2)(1 - \eta y(z)^2)$$

($\xi = 1,\ \eta = k^2$) with the initial data $y(0) = 0,\ y'(0) = 1$. The functions $y(z) = \operatorname{sn}(z,k)/\operatorname{cn}(z,k)$ ($\xi = -1,\ \eta = k^2 - 1$) and $y(z) = \operatorname{sn}(z,k)/\operatorname{dn}(z,k)$ ($\xi = -k^2,\ \eta = 1 - k^2$) satisfy similar differential relations (with the same initial data). In terms of the formal group law (1.5), the properties (of elliptic functions of level 2) listed above can be reformulated as follows.

Theorem 2. *Any elliptic function of level* 2 *is the exponential of the formal group* (1.5) *in which* $A_2(t) = 1$ *and, for* $B_1 = 0$*, the function* $B(t) = B_2(t)$ *satisfies the quadratic relation*

$$B(t)^2 = 1 + 2B_2 t^2 + (B_2^2 + 2B_4)t^4. \tag{1.11}$$

The coefficient ring of the formal group law (1.10) is described in [8].

For the level $n = 3$, the following result was proved in [7] and [9].

Theorem 3. *An arbitrary elliptic function of level* 3 *is the exponential of a formal group* (1.5) *in which, for* $B_1 = 2A_1$ *and* $A_2 = 0$*, the series* $A(t) = A_3(t)$ *and* $B(t) = B_3(t)$ *satisfy the relations*

$$A(t)^2 = B(t), \tag{1.12}$$

$$A(t)^3 - 3A_1 t A(t) = 1 + (A_1^3 + 3A_3)t^3, \tag{1.13}$$

$$B(t)(B(t) - 3A_1 t)^2 = (1 + (A_1^3 + 3A_3)t^3)^2. \tag{1.14}$$

In this theorem, the first relation connects $A(t)$ and $B(t)$, the second one connects t and $A(t)$, and the third one t and $B(t)$. (The last formula becomes somewhat more complicated if the coefficients are expressed in terms of $B_1 = 2A_1$ and $B_3 = 2A_3$.) Note that a formal group with the relation $A(t)^2 = B(t)$ occurred in [10] in the study of formal groups obtained from the addition law for points on an elliptic curve.

For the level $n = 4$, as was proved in [11], for $A_2 = B_1 = 0$, the series $A(t) = A_4(t)$ and $B(t) = B_4(t)$ are connected by the relation

$$(2B(t) + 3A_1 t)^2 = 4A(t)^3 - (3A_1^2 - 8B_2)t^2 A(t)^2. \tag{1.15}$$

Seemingly, this relation is one of the simplest formulas connecting t, $A(t)$, and $B(t)$.

As is well known, every two elliptic functions with the same periods are connected by an algebraic relation (see [1, Sec. 16]). For an arbitrary level n, the functions f_n^n, a_n^n, and b_n^n are elliptic (see Lemma 2 below) and, therefore, every two of the three functions f_n, a_n, and b_n must be connected by an algebraic relation. Hence, for every $n \geq 2$, every two of the three power series t, $A_n(t)$, $B_n(t)$ must also be connected by an algebraic relation similar to (1.12)–(1.14). In the present paper, we solve the problem of finding these relations in the case of $n = 4$. The main idea is to express the functions a_n, b_n, and f_n using the functions $\wp$ and $\wp'$ associated with the original lattice Γ and then eliminate $\wp$ and $\wp'$ from the relations thus obtained with the help of the following identity ($g_2 = g_2(\Gamma)$, $g_3 = g_3(\Gamma)$):

$$\wp'(x)^2 = 4\wp(x)^3 - g_2\wp(x) - g_3. \tag{1.16}$$

In Secs. 3 and 4, following this approach, we prove Theorems 2 and 3, respectively.

In Sec. 5, a solution of the problem posed above is given.

Theorem 4. *An arbitrary elliptic function of level* 4 *is the exponential of a formal group* (1.5) *in which, for $B_1 = A_1/2$, the series $A(t) = A_4(t)$ and $B(t) = B_4(t)$ satisfy the relations*

$$C_3A(t)^4(A(t)^2-1)^2 + C_1^3B(t)^2(A(t)^3 - B(t)^2) = 0, \tag{1.17}$$

$$(A(t)^2-1)^2 = C_1^2t^2A(t) + C_4t^4, \tag{1.18}$$

$$C_1^2B(t)(B(t)+C_1t)^3 - C_4t^2B(t)(2B(t)+C_1t) = (1-C_4t^4)(C_1^2 - C_3^2t^4), \tag{1.19}$$

where $C_1 = -2A_1$, $C_3 = -A_1^3/2 - 4A_3$, and $C_4 = C_1C_3 = A_1^4 + 8A_1A_3$.

Note that relation (1.15) does not follow from (1.17)–(1.19), because it is obtained under the assumption that $B_1 = 0$ rather than $B_1 = A_1/2$.

It follows from the definitions in (1.1), (1.3), and (1.4) that, for an arbitrary z, the functions a/f^2 and b/f are elliptic, and thus are connected by an algebraic relation. Therefore, for the universal Buchstaber formal group (1.5), there is an algebraic relation connecting t, $A(t)$, and $B(t)$. It is established in Sec. 6.

Theorem 5. *The series $A(t)$ and $B(t)$ defining a formal group law* (1.5) *over a torsion-free group are related for $A_1 = 2B_1$ by an algebraic relation*

$$\begin{aligned} &A(t)(A(t)-B(t)^2+(B_1^2+2B_2)t^2) \\ &\quad + t^3(B(t)(2B_1B_2+3A_3)+t(2B_4+2B_1^2B_2+9B_1A_3+B_2^2)) = 0. \end{aligned} \tag{1.20}$$

2. AUXILIARY ASSERTIONS

Lemma 1. *The points of orders* 2 *and* 4 *are characterized by the conditions $\wp'(z_2) = 0$ and $\wp'(2z_4) = 0$, respectively. The points of order $n \geq 3$ $(n \neq 4)$ are characterized by the conditions*

$$\wp((n-1)z_n) = \wp(z_n) \neq \infty, \qquad \wp(kz_n) \neq \wp(z_n) \quad \text{for} \quad 2 \leq k \leq n-3. \tag{2.1}$$

Proof. The mapping $z \to (\wp(z), \wp'(z))$ defines an isomorphism between the torus $\mathbb{C}/\Gamma$ and the elliptic curve $y^2 = 4x^3 - g_2(\Gamma)x - g_3(\Gamma)$. Hence the points of order n can be characterized by the conditions

$$\wp((n-1)z_n) = \wp(z_n), \qquad \wp'((n-1)z_n) = -\wp'(z_n) \tag{2.2}$$

under the assumption that n cannot be replaced by a lesser number. Therefore, the points of order 2 are characterized by the condition $\wp'(z_2) = 0$. Thus, the points of order 4 are characterized by the condition $\wp'(2z_4) = 0$.

Let $n \geq 3$ and $n \neq 4$. Obviously, conditions (2.1) are *necessary* for z_n to be a point of order n. Let us verify their *sufficiency*. If the second condition in (2.2) is violated, then

$$\wp((n-1)z_n) = \wp(z_n), \qquad \wp'((n-1)z_n) = \wp'(z_n).$$

Thus, we have $(n-1)z_n \equiv z_n \pmod{\Gamma}$. For $n = 3$, this congruence contradicts the condition that $\wp(z_n) \neq \infty$. For $n \geq 5$, the congruence thus obtained means that z_n is a point of order $d \mid (n-2)$. However, in this case $\wp(kz_n) = \wp(z_n)$, where $k = 3$ for $d = 2$ and $k = d-1 \leq n-3$ for $d > 2$. The contradiction shows that conditions (2.2) indeed follow from (2.1). Moreover, in relations (2.2), one cannot replace n by a lesser number (a divisor of n), because this also contradicts the second condition in (2.1). □

Lemma 2. *Let $n \geq 2$ and let the functions f_n, a_n, and b_n be given by relations* (1.6), (1.8), *and* (1.9), *respectively. Then*

$$f_n(x)^n = \frac{\sigma(x)^n\sigma(z_n)^{n-1}\sigma((n-1)z_n)}{\sigma(z_n-x)^{n-1}\sigma(x+(n-1)z_n)}.$$

The functions f_n^n, a_n^n, and b_n^n are elliptic.

Proof. Using the formula (see [12, 18.2.20])

$$\sigma(t+2k\omega_1+2l\omega_2)=(-1)^{k+l+kl}\sigma(t)e^{(t+k\omega_1+l\omega_2)(2k\eta_1+2l\eta_2)}$$

for $t=x-z_n$ and $t=-z_n$, we see that

$$\frac{\sigma(x+(n-1)z_n)\sigma(-z_n)}{\sigma((n-1)z_n)\sigma(x-z_n)}=e^{nh_nx}.$$

Hence

$$f_n(x)^n=\frac{\sigma(x)^n\sigma(z_n)^n}{\sigma(z_n-x)^n}e^{-nh_nx}=\frac{\sigma(x)^n\sigma(z_n)^{n-1}\sigma((n-1)z_n)}{\sigma(z_n-x)^{n-1}\sigma(x+(n-1)z_n)}.$$

The definition (1.6) implies the equalities

$$f_n(x+2\omega_1)=f_n(x)e^{2\pi il/n},\qquad f_n(x+2\omega_2)=f_n(x)e^{-2\pi ik/n},$$

and therefore f_n^n is an elliptic function. The ellipticity of the function a_n^n follows from the definition of a_n and from the ellipticity of f_n^n.

It follows from the standard formulas

$$\zeta(u+v)=\zeta(u)+\zeta(v)+\frac{1}{2}\cdot\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)} \tag{2.3}$$

(see [12, 18.4.3]) and $\zeta(z)=(\log z)'$ that

$$\frac{f'(x)}{f(x)}=-\frac{1}{2}\cdot\frac{\wp'(x)+\wp'(z)}{\wp(x)-\wp(z)}+\alpha.$$

Thus, the logarithmic derivative of the function f is an elliptic function. Hence the function

$$\frac{b(x)}{f(x)}=\frac{f'(x)}{f(x)}+\mu=-\frac{1}{2}\cdot\frac{\wp'(x)+\wp'(z)}{\wp(x)-\wp(z)}+\alpha+\mu \tag{2.4}$$

is also elliptic. Therefore, the ellipticity of b_n^n follows from the ellipticity of f_n^n and the ellipticity of the ratio b_n/f_n. $\square$

Lemma 3. *Let $n\geq 2$. Then the quantity α_n, defined by relation* (1.7) *can be expressed as*

$$\alpha_n=\frac{(n-1)\zeta(z_n)-\zeta((n-1)z_n)}{n}.$$

Proof. Using the formula (see [12, 18.2.19])

$$\zeta(t+2k\omega_1+2l\omega_2)=\zeta(t)+2k\eta_1+2l\eta_2,$$

for $t=-z_n$, we see that $nh_n=\zeta(z_n)+\zeta((n-1)z_n)$. Substituting this expression into the definition of the quantity α_n, we arrive at the assertion of the lemma. $\square$

The following assertion can be verified by immediate calculations.

Lemma 4. *The initial coefficients of the series $A(t)=a(f^{-1}(t))$ and $B(t)=b(f^{-1}(t))$, where the functions f, a, and b are defined by the relations* (1.1), (1.4), *and* (1.3), *respectively, have the form*

$$A_1=2\alpha,\qquad A_2=0,\qquad A_3=\frac{1}{3}(\alpha^3-3\alpha\wp(z)-\wp'(z)),\qquad A_4=-A_1A_3, \tag{2.5}$$

$$B_1=2\alpha+\mu,\qquad B_2=\frac{1}{2}(3\wp(z)-\alpha^2),\qquad B_3=2A_3, \tag{2.6}$$

$$B_4=\frac{1}{8}(-9\alpha^4+30\alpha^2\wp(z)+3\wp(z)^2+12\alpha\wp'(z)-g_2). \tag{2.7}$$

3. LEVEL 2

Proof of Theorem 2. Applying in succession Lemma 2 and the formula (see [12, 18.4.4])

$$\wp(u)-\wp(v)=-\frac{\sigma(u-v)\sigma(u+v)}{\sigma(u)^2\sigma(v)^2}, \tag{3.1}$$

we see that

$$\frac{1}{f_2(x)^2}=-\frac{\sigma(x-z_2)\sigma(x+z_2)}{\sigma(x)^2\sigma(z_2)^2}=\wp(x)-\wp(z_2). \tag{3.2}$$

Thus, by the definition in (1.4), $a_2(t)=1$. Hence $A_2(t)=1$.

Let us prove the validity of the relation

$$b_2(x)^2=1+3\wp(z_2)f_2(x)^2+\frac{\wp''(z_2)}{2}f_2(x)^4. \tag{3.3}$$

By Lemma 3, $\alpha_2=0$. By the assumption $B_1=0$ and by the relation $B_1=2\alpha+\mu$, we must choose $\mu_2=0$ in the definition of the function $b_2(x)$. It follows from the condition $\wp'(z_2)=0$ characterizing points of second order (see Lemma 1) and from relation (2.4) that

$$\frac{b_2(x)}{f_2(x)}=-\frac{1}{2}\cdot\frac{\wp'(x)}{\wp(x)-\wp(z_2)}. \tag{3.4}$$

Expressing $\wp(x)$ and $\wp'(x)$ using relations (3.2) and (3.4), substituting the expressions into the Weierstrass equation (1.6), and taking into account the condition $4\wp(z_2)^3-g_2\wp(z_2)-g_3=0$, we arrive at the relation (3.3). It follows from formulas (2.6) and (2.7) for $\alpha=\mu=0$ that

$$B_2=\frac{3}{2}\wp(z_2),\qquad 2B_4=\frac{\wp''(z_2)}{2}-B_2^2.$$

Therefore, when passing to the series $B_2(t)=b_2(f_2^{-1}(t))$, relation (3.3) becomes (1.11). □

Remark 1. Theorem 2 claims, in particular, that the condition $A(t)=1$ is necessary for the exponential of Buchstaber's formal group to be an elliptic function of level 2. It follows from relations (2.5) that this condition is also sufficient. Indeed, if $A_1=A_3=0$, then $\alpha=0$ and $\wp'(z)=0$, i.e., z is a point of order 2.

4. LEVEL 3

Lemma 5. *The following equalities hold*:

$$\frac{a_3(x)^2}{f_3(x)}=-\frac{1}{2}\left(\frac{\wp'(x)+\wp'(z_3)}{\wp(x)-\wp(z_3)}+\frac{\wp''(z_3)}{\wp'(z_3)}\right), \tag{4.1}$$

$$f_3(x)^3=-\frac{1}{2}\cdot\frac{1}{(\wp(x)-\wp(z_3))^2}\left(\frac{\wp'(x)+\wp'(z_3)}{\wp(x)-\wp(z_3)}+\frac{\wp''(z_3)}{\wp'(z_3)}\right). \tag{4.2}$$

Proof. It suffices to prove only the first equality, since the other one is a consequence of the first one and of the definition of the function $a_3(x)$. By Lemma 2,

$$f_3(x)^3=\frac{\sigma(x)^3\sigma(z_3)^2\sigma(2z_3)}{\sigma(z_3-x)^2\sigma(x+2z_3)}.$$

It follows from (3.1) that

$$(\wp(x)-\wp(z))^2=\frac{\sigma(x-z)^2\sigma(x+z)^2}{\sigma(x)^4\sigma(z)^4}. \tag{4.3}$$

Thus,

$$\frac{a_3(x)^2}{f_3(x)}=f_3(x)^3(\wp(x)-\wp(z_3))^2=\frac{\sigma(2z_3)\sigma(x+z_3)^2}{\sigma(x)\sigma(z_3)^2\sigma(x+2z_3)}=\frac{-\sigma(2z_3)}{\sigma(z_3)^4(\wp(x+z_3)-\wp(z_3))}.$$

It follows from the relation $\sigma(2z) = -\wp'(z)\sigma(z)^4$ (see [12, 18.4.8]) that

$$\frac{a_3(x)^2}{f_3(x)} = \frac{\wp'(z_3)}{\wp(x+z_3) - \wp(z_3)}\,.$$

To prove the lemma, it remains to prove the validity of the equality

$$\frac{\wp'(z)}{\wp(x+z) - \wp(z)} = -\frac{1}{2}\cdot\frac{\wp'(x) - \wp'(2z)}{\wp(x) - \wp(2z)} - \frac{1}{2}\cdot\frac{\wp''(z)}{\wp'(z)}\,. \tag{4.4}$$

Substituting the values $u = v = z$ and $u = -x$, $v = 2z + x$ into (2.3), we see that

$$\frac{\wp''(z)}{2\wp'(z)} = \zeta(2z) - 2\zeta(z), \tag{4.5}$$

$$-\frac{1}{2}\cdot\frac{\wp'(x) - \wp'(2z)}{\wp(x) - \wp(2z)} = \zeta(x) + \zeta(2z) - \zeta(x+2z). \tag{4.6}$$

Adding the formulas obtained by the substituting the values $u = x + z$, $v = z$ and $u = -x - z$, $v = z$ into (2.3), we obtain an additional equality,

$$\frac{\wp'(z)}{\wp(x+z) - \wp(z)} = 2\zeta(z) - \zeta(x+2z) - \zeta(-x). \tag{4.7}$$

The desired formula (4.4) is the sum of (4.5), (4.6), and (4.7). □

In what follows, we use the notation $L_n = \wp''(z_n)/\wp'(z_n)$.

Proof of Theorem 3. Let us prove first that an arbitrary elliptic function of level 3 satisfies the additivity theorem (1.2), in which, for $\mu_3 = -L_3/3$ and for the functions $f(x) = f_3(x)$, $a(x) = a_3(x)$, and $b(x) = b_3(x)$, the following equalities hold:

$$a(x)^2 = b(x), \tag{4.8}$$

$$a(x)^3 + L_3 a(x) f(x) = 1 - \wp'(z_3) f(x)^3, \tag{4.9}$$

$$b(x)(b(x) + L_3 f(x))^2 = (1 - \wp'(z_3) f(x)^3)^2. \tag{4.10}$$

Using Lemma 3 and the formula (see [12, 18.4.7])

$$\zeta(2z) = 2\zeta(z) + \frac{1}{2}\cdot\frac{\wp''(z)}{\wp'(z)}\,,$$

we see that

$$\alpha_3 = \frac{2\zeta(z_3) - \zeta(2z_3)}{3} = -\frac{1}{6}\cdot\frac{\wp''(z_3)}{\wp'(z_3)} = -\frac{L_3}{6}\,, \qquad \alpha_3 + \mu_3 = -\frac{L_3}{2}\,.$$

Therefore, relation (4.8) follows from (2.4) and (4.1).

It follows from the duplication formula (see [12, 18.4.5])

$$\wp(2z) = -2\wp(z) + \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2 \tag{4.11}$$

that the condition $\wp(2z_3) = \wp(z_3)$ characterizing points of third order (see Lemma 1) can be represented in the form

$$12\wp(z_3)\wp'(z_3)^2 = \wp''(z_3)^2. \tag{4.12}$$

Expressing $\wp(x)$ and $\wp'(x)$ from relations (4.2) and (1.4), substituting the obtained expressions into the Weierstrass equation (1.16), and taking into account the conditions

$$\wp'(z_3)^2 = 4\wp(z_3)^3 - g_2\wp(z_3) - g_3 = 0$$

and (4.12), we obtain (4.9). Relations (2.4) and $b(x) = f(x)^4(\wp(x) - \wp(z))^2$ imply (4.10) in the same way.

We see from (2.5) that, for $\alpha_3 = -L_3/6$, the relations $L_3 = 3A_1$ and $\wp'(z) = -A_1^3 - 3A_3$ hold. Therefore, when passing to the series $A_3(t) = a_3(f_3^{-1}(t))$ and $B_3(t) = b_3(f_3^{-1}(t))$, relations (4.9) and (4.10) acquire the form (1.13) and (1.14), respectively. Equation (1.12) is an immediate consequence of (4.8). □

Remark 2. As was proved in [6], the relation $A(t)^2 = B(t)$ is not only a necessary but also a sufficient condition for the exponential of the Buchstaber formal group (1.5) to be an elliptic function of level 3. This can readily be verified using relations (2.5)–(2.7). Equating to zero the coefficients at t^2 and t^4 in the formula $A(t)^2 - B(t) = 0$, we obtain the relations $\wp(z) = 3\alpha^2$ and $\wp''(z) + 6\alpha\wp'(z) = 0$, which imply that the point z satisfies the condition (4.12) characterizing the points of order three.

5. LEVEL 4

Lemma 6. *Let z_4 be a point of order 4. Then*

$$a_4(x)^2 = \frac{\wp(x - z_4) - \wp(2z_4)}{\wp(z_4) - \wp(2z_4)}, \tag{5.1}$$

$$f_4(x)^4 = \frac{1}{(\wp(x) - \wp(z_4))^2} \cdot \frac{\wp(x - z_4) - \wp(2z_4)}{\wp(z_4) - \wp(2z_4)}. \tag{5.2}$$

Proof. As in the case $n = 3$, we restrict ourselves to the proof of the first relation, because the second relation is a consequence of the first one and of the definition of the function $a(x)$. By Lemma 2,

$$f_4(x)^4 = \frac{\sigma(x)^4\sigma(z_4)^3\sigma(3z_4)}{\sigma(z_4 - x)^3\sigma(x + 3z_4)}.$$

Using formula (4.3), we see that

$$a_4(x)^2 = f_4(x)^4(\wp(x) - \wp(z_4))^2 = \frac{\sigma(3z_4)\sigma(x + z_4)^2}{\sigma(z_4)\sigma(x + 3z_4)\sigma(z_4 - x)}.$$

Applying formula (3.1) to both sides of the equality $\wp(z_4) - \wp(x) = \wp(3z_4) - \wp(x)$, we see that

$$\frac{\sigma(3z_4)\sigma(x + z_4)^2}{\sigma(z_4)\sigma(x + 3z_4)\sigma(z_4 - x)} = -\frac{\sigma(x - 3z_4)\sigma(x + z_4)\sigma(z_4)}{\sigma(x - z_4)^2\sigma(3z_4)}.$$

Thus,

$$a_4(x)^2 = -\frac{\sigma(x - 3z_4)\sigma(x + z_4)\sigma(z_4)}{\sigma(x - z_4)^2\sigma(3z_4)} = \frac{\wp(x - z_4) - \wp(2z_4)}{\wp(z_4) - \wp(2z_4)}. \quad \square$$

Lemma 7. *We have*

$$\alpha_4 = -\frac{1}{4} \cdot \frac{\wp''(z_4)}{\wp'(z_4)} = \frac{\wp'(z_4)^3}{\wp''(z_4)^2 - \wp'(z_4)\wp'''(z_4)}.$$

Proof. By Lemma 3,

$$\alpha_4 = \frac{3\zeta(z_4) - \zeta(3z_4)}{4}.$$

Using the identity (see [12, 18.4.9])

$$\zeta(3z) = 3\zeta(z) + \frac{4\wp'(z)^3}{\wp'(z)\wp'''(z) - \wp''(z)^2}$$

and relation (5.7), we obtain the desired representation for α_4. □

Proof of Theorem 4. Let us prove first that, for $\mu = -L_4/4$, the functions $f(x) = f_4(x)$, $a(t) = a_4(t)$, and $b(t) = b_4(t)$ satisfy the conditions

$$\wp'(z)a(t)^4(a(t)^2-1)^2 + L_4^3 b(t)^2(a(t)^3 - b(t)^2) = 0, \tag{5.3}$$

$$(a(t)^2-1)^2 = L_4^2 f(t)^2 a(t) + \wp''(z) f(t)^4, \tag{5.4}$$

$$\begin{aligned} &b(t)L_4^2(b(t)+L_4 f(t))^3 - b(t)f(t)^2 L_4 \wp'(z_4)(2b(t)+f(t)L_4) \\ &\quad = (1-\wp''(z)f(t)^4)(L_4^2 - \wp'(z_4)^2 f(t)^4). \end{aligned} \tag{5.5}$$

It follows from the formula (see [12, 18.4.6])

$$\wp'(2z) = \frac{-4\wp'(z)^4 + 12\wp(z)\wp'(z)^2\wp''(z) - \wp''(z)^3}{4\wp'(z)^3}$$

that the condition $\wp'(2z_4) = 0$ characterizing the points of order 4 (see Lemma 1) can be represented in the form

$$12\wp(z_4)\wp'(z_4)^2\wp''(z_4) = 4\wp'(z_4)^4 + \wp''(z_4)^3. \tag{5.6}$$

Since $12\wp(z)\wp'(z) = \wp'''(z)$, one can also represent this relation in the form

$$\wp'(z_4)\wp''(z_4)\wp'''(z_4) = 4\wp'(z_4)^4 + \wp''(z_4)^3. \tag{5.7}$$

It follows from (4.11) that formula (5.7) is equivalent to the equality

$$\wp(2z_4) - \wp(z_4) = -\frac{\wp'(z_4)^2}{\wp''(z_4)}. \tag{5.8}$$

It follows from formulas (5.6)–(5.8) that all the parameters that we need can be expressed using $\wp'(z_4)$ and $\wp''(z_4)$; namely,

$$\wp(z_4) = \frac{4\wp'(z_4)^4 + \wp''(z_4)^3}{12\wp'(z_4)^2\wp''(z_4)}, \qquad \wp(2z_4) = \wp(z_4) - \frac{\wp'(z_4)^2}{\wp''(z_4)},$$

$$g_2 = 12\wp(z_4)^2 - 2\wp''(z_4), \qquad g_3 = 4\wp(z_4)^3 - g_2\wp(z_4) - \wp'(z_4)^2.$$

To prove relation (5.4), it remains to apply to $\wp(x - z_4)$ the additivity theorem (see [12, 18.4.1]):

$$\wp(u+v) + \wp(u) + \wp(v) = \frac{1}{4}\left(\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\right)^2,$$

express the quantities $\wp(x)$ and $\wp'(x)$ from relations (1.4) and (5.1), and substitute the expressions into the Weierstrass relation (1.16).

For $z = z_4$ and $\mu = \alpha_4 = -L_4/4$, relation (2.4) becomes

$$\frac{b_4(x)}{f_4(x)} = -\frac{1}{2}\left(\frac{\wp'(x) + \wp'(z_4)}{\wp(x) - \wp(z_4)} + \frac{\wp''(z_4)}{\wp'(z_4)}\right).$$

Eliminating the quantities $\wp(x)$ and $\wp'(x)$ from this relation and from (5.2), we arrive at relation (5.5) connecting $b_4(x)$ and $f_4(x)$.

It follows from (2.5) and (1.4) that

$$\frac{b_4(x)^2}{a_4(x)} = \frac{b_4(x)^2}{f_4(x)^2} \cdot \frac{f_4(x)^2}{a_4(x)} = \frac{1}{4(\wp(x) - \wp(z_4))}\left(\frac{\wp'(x) + \wp'(z_4)}{\wp(x) - \wp(z_4)} + \frac{\wp''(z_4)}{\wp'(z_4)}\right)^2.$$

Eliminating the quantities $\wp(x)$ and $\wp'(x)$ from this relation and from (5.1), we arrive at relation (5.3) connecting $a_4(x)$ and $b_4(x)$.

We see from formula (2.5) that, for $\mu = \alpha_4 = -L_4/4$, the following relations hold:

$$L_4 = -2A_1, \qquad B_1 = \frac{A_1}{2}, \qquad \wp'(z_4) = -\frac{A_1^3}{2} - 4A_3, \qquad \wp''(z_4) = A_1^4 + 8A_1A_3.$$

Therefore, when passing to the series $A_4(t) = a_4(f_4^{-1}(t))$ and $B_4(t) = b_4(f_4^{-1}(t))$, we see that relations (5.3)–(5.5) take the form (1.17)–(1.19), respectively. □

Remark 3. As was noted above, in the cases $n = 2$ and $n = 3$, the relations $A(t) = 1$ and $A(t)^2 = B(t)$ are not only necessary but also sufficient conditions for the exponential of the Buchstaber formal group (1.5) to be an elliptic function of level 2 or 3, respectively (see Remarks 1 and 2). Let us consider relation (1.17) connecting the series $A(t)$ and $B(t)$ for $n = 4$. This relation becomes a sufficient condition for the exponential of the Buchstaber formal group (1.5) to be an elliptic function of level 4 if we assume in addition that $A(t) \neq 1$. Indeed, if $A_1 = 2\alpha = 0$ and $A_3 = 0$, then f is an elliptic function of level 2 and $A(t) = 1$. If $A_1 = 2\alpha = 0$ and $A_3 \neq 0$, then $C_3 \neq 0$, and we again see from relation (1.17) that $A(t) = 1$. If $A_1 = 2\alpha \neq 0$, then, using formulas (2.5)–(2.7) and equating to zero the coefficients at t, t^2, and t^4 in relation (1.17), we see that

$$\mu = \alpha, \qquad 16\alpha^3 - \wp'(z) - 12\alpha\wp(z) = 0, \qquad \alpha = -\frac{\wp''(z)}{4\wp'(z)}.$$

The last two relations imply the validity of condition (5.6) characterizing the points of order 4.

6. ALGEBRAIC RELATION FOR THE UNIVERSAL BUCHSTABER FORMAL GROUP

Proof of Theorem 5. Let us prove first that, for $\mu = -\alpha$, the functions $a(x)$ and $b(x)$ are connected by the relation

$$a(x)(a(x) - b(x)^2 + 3\wp(z)f(t)^2) = f(x)^3\left(b(x)\wp'(z) - \frac{\wp''(z)}{2}f(x)\right). \tag{6.1}$$

Let us express the values $\wp(x)$ and $\wp'(x)$ from relations (1.4) and (2.4):

$$\wp(x) = \frac{a(x)}{f(x)^2} + \wp(z),$$
$$\wp'(x) = -\frac{2a(x)b(x)}{f(x)^3} - \wp'(z).$$

Substituting these expressions into the Weierstrass equations (1.16), we see that the parameter g_3 is cancelled. Replacing g_2 by the formula $g_2 = 12\wp(z)^2 - 2\wp''(z)$, we arrive at the relation (6.1).

It follows from formulas (2.5)–(2.7) that, for $\mu = -\alpha$, the following relations hold:

$$3\wp(z) = B_1^2 + 2B_2, \qquad \wp'(z) = -2B_1B_2 - 3A_3,$$
$$\frac{\wp''(z)}{2} = 2B_4 + 2B_1^2B_2 + 9B_1A_3 + B_2^2.$$

Therefore, when passing to the series $A(t) = a(f^{-1}(t))$ and $B(t) = b(f^{-1}(t))$, relation (6.1) acquires the form (1.20). □

Remark 4. Formulas (1.20) and (6.1), which are proved above for $B_1 = A_1/2$ ($\mu = -\alpha$), cannot be used simultaneously with the relations (1.12)–(1.14) and (1.17)–(1.19), obtained for the levels $n = 3$ ($B_1 = 2A_1$, $\mu = 2\alpha$) and $n = 4$ ($2B_1 = 3A_1$, $\mu = \alpha$), respectively. For $n = 2$, no contradictions occur, because the relations $A_2(t) = 1$ and $B_2(t)^2 = 1 + 2B_2t^2 + (B_2^2 + 2B_4)t^4$ are obtained under the assumptions $\mu = \alpha_2 = 0$, which do not contradict the equality $\mu = -\alpha$.

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